Mechanics of Materials

Chapter 3

Torsion

3.1 *Introduction*

- In many engineering applications, members are required to carry **torsional loads**.
- Consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads. Circular shafts are commonly used to transmit power in rotating machinery.
- Also discuss another important application—torsion of thin-walled tubes..

3.1 *Torsion of Circular Shafts*

a. Simplifying assumptions

- During the deformation, the cross sections are not distorted in any manner — they remain plane, and the radius r does not change. In addition, the length L of the shaft remains constant.

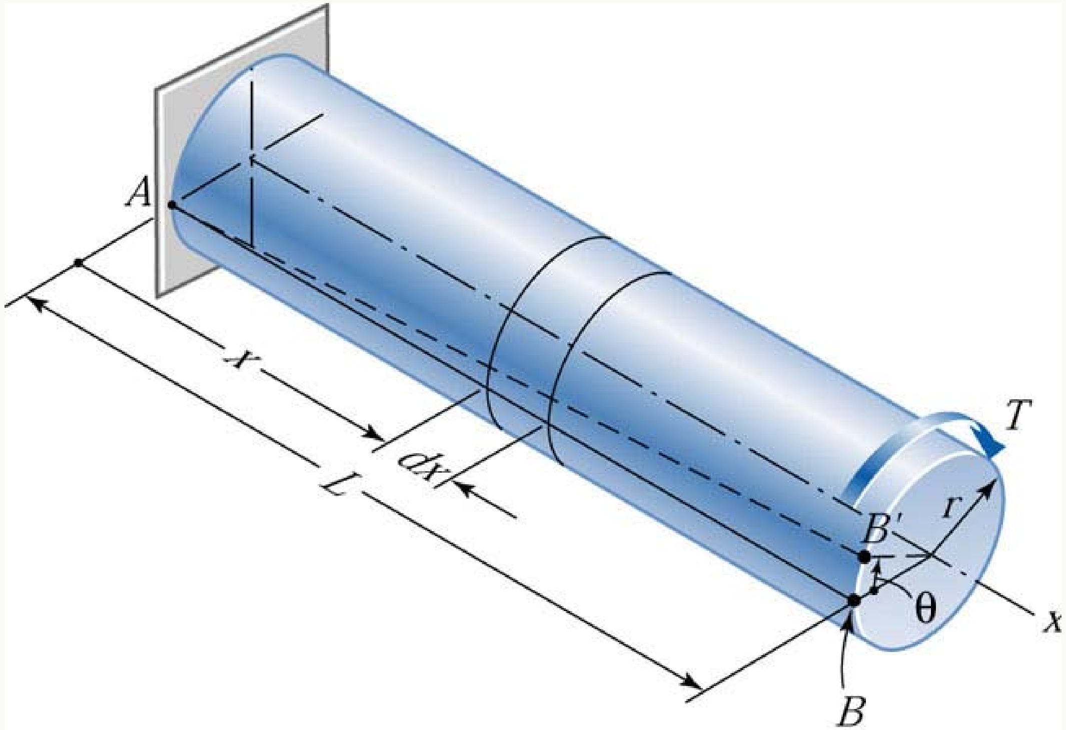
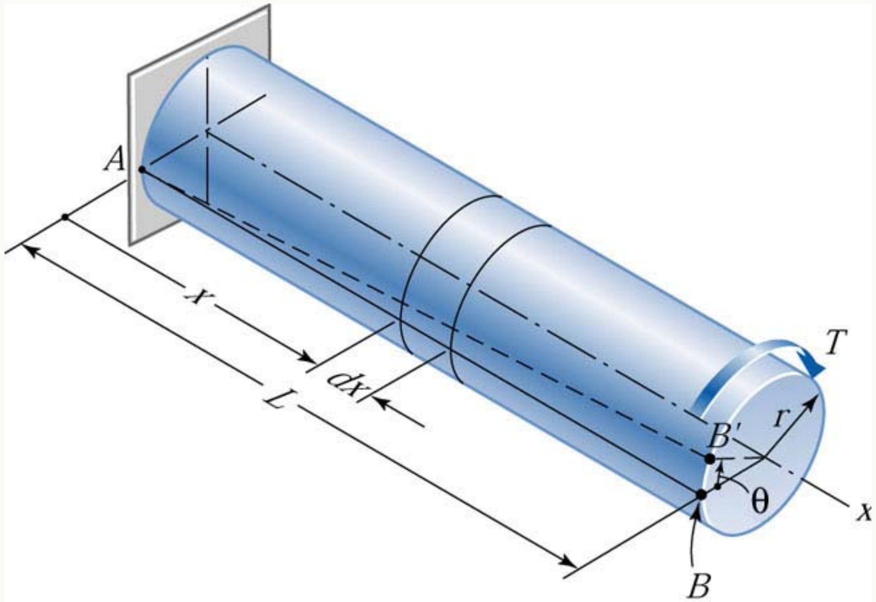


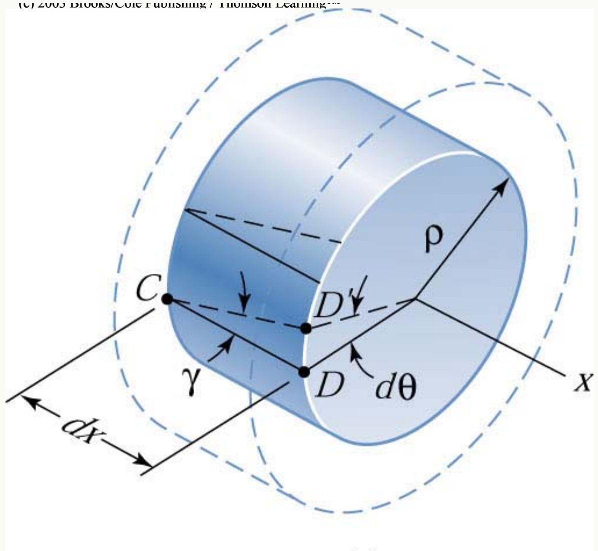
Figure 3.1 Deformation of a circular shaft caused by the torque *T*. The initially straight line *AB* deforms into a helix.

- Based on these observations, we make the following assumptions:
  - Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.
  - Cross sections do not deform (there is no strain in the plane of the cross section).
  - The distances between cross sections do not change (the axial normal strain is zero).
- *Each cross section rotates as a rigid entity about the axis of the shaft*. Although this conclusion is based on the observed deformation of a cylindrical shaft carrying a **constant** internal torque, we assume that the result remains valid even if the diameter of the shaft or the internal torque **varies** along the length of the shaft.

b. *Compatibility*

- Because the cross sections are separated by an infinitesimal distance, the difference in their rotations, denoted by the angle $d\theta$, is also infinitesimal.
- As the cross sections undergo the relative rotation $d\theta$, *CD* deforms into the helix *CD*. By observing the distortion of the shaded element, we recognize that the helix angle γ is the *shear strain* of the element.





(a)

From the geometry of Fig.3.2(a), we obtain DD´ $= \rho\, d\theta = \gamma\, dx$, from which the shear strain γ is

$$\gamma = \frac{d\theta}{dx}\rho \qquad (3.1)$$

The quantity $d\theta/dx$ is the *angle of twist per unit length,* where θ is expressed in radians. The corresponding shear stress, illustrated in Fig. 3.2 (b), is determined from Hooke´s law:

$$\tau = G\gamma = G\frac{d\theta}{dx}\rho \qquad (3.2)$$

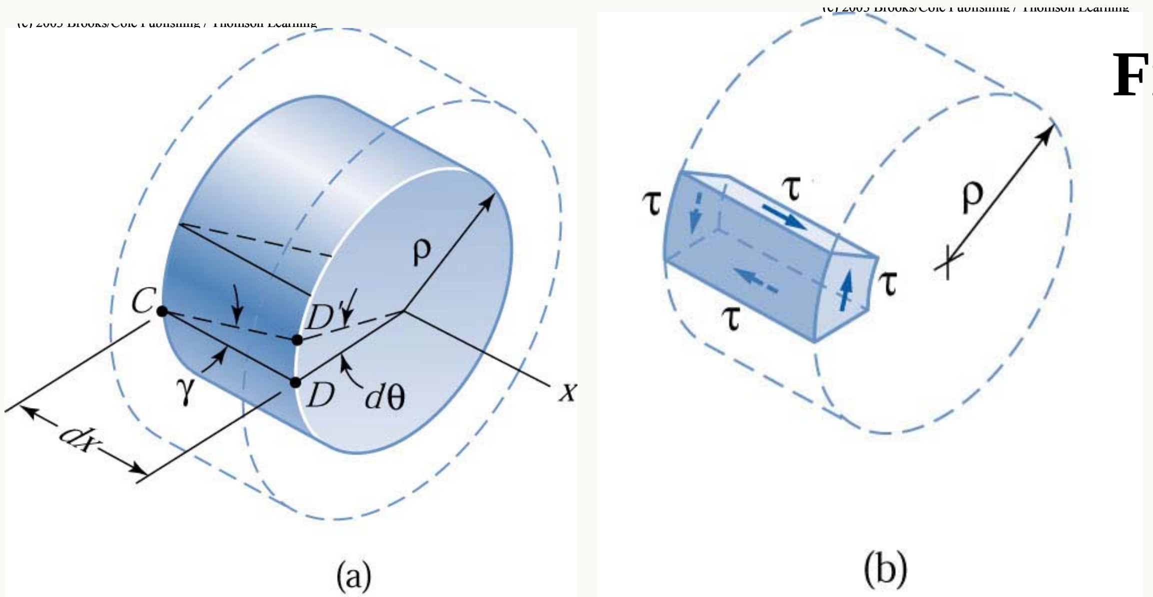


Figure 3.2 (a) Shear strain of a material element caused by twisting of the shaft; (b) the corresponding shear stress.

- *the shear stress varies linearly with the radial distance ρ from the axial of the shaft.* $\tau = G\gamma = G\frac{d\theta}{dx}\rho$
- The variation of the shear stress acting on the cross section is illustrated in Fig. 3.3. The maximum shear stress, denoted by τ_{max}, occurs at the surface of the shaft.
- Note that the above derivations assume neither a constant internal torque nor a constant cross section along the length of the shaft.

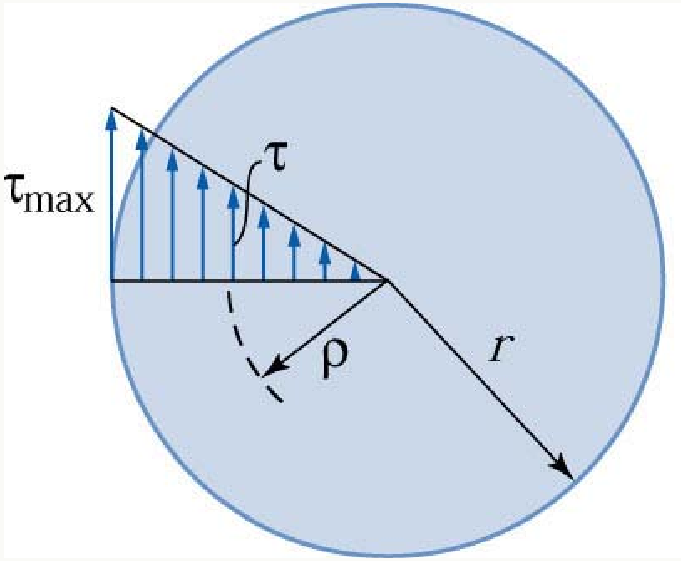


Figure 3.3 Distribution of shear stress along the radius of a circular shaft.

c. *Equilibrium*

- Fig. 3.4 shows a cross section of the shaft containing a differential element of area *dA* loaded at the radial distance ρ from the axis of the shaft.

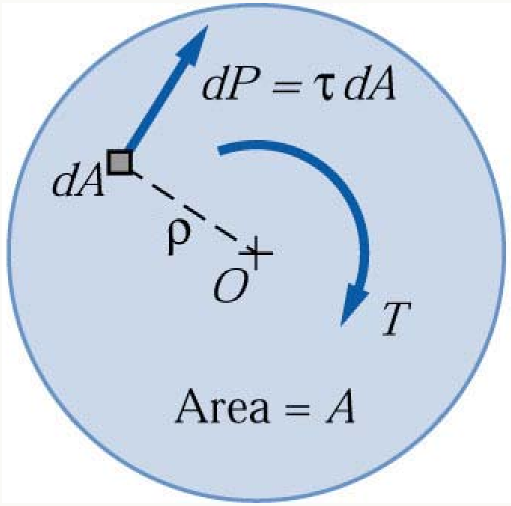


Figure 3.4 Calculating the Resultant of the shear stress acting on the cross section. Resultant is a couple equal to the internal torque *T*.

- The shear force acting on this area is $dP = \tau\, dA = G\,(d\theta/dx)\,\rho\, dA$, directed perpendicular to the radius. Hence, the moment (torque) of dP about the center O is $\rho\, dP = G\,(d\theta/dx)\,\rho^2 dA$. Summing the contributions and equating the result to the internal torque yields $\int_A \rho dP = T$, or

$$G\frac{d\theta}{dx}\int_A \rho^2 dA = T$$

Recognizing that is the polar moment of inertia of the cross-sectional area, we can write this equation as $G\,(d\theta/dx)\,J = T$, or

$$\frac{d\theta}{dx} = \frac{T}{GJ} \qquad (3.3)$$

The rotation of the cross section at the free end of the shaft, called **the angle of twist** θ , is obtained by integration:

$$\theta = \int_o^L d\theta = \int_o^L \frac{T}{GJ}dx \qquad (3.4a)$$

As in the case of a **prismatic bar** carrying a constant torque, then reduces **the torque-twist relationship**

$$\theta = \frac{TL}{GJ} \qquad (3.4b)$$

Note the similarity between Eqs. (3.4) and the corresponding formulas for axial deformation: $\delta = \int_o^L (P/EA)dx$ and $\delta = PL/(EA)$

Notes on the Computation of angle of Twist

- 1.In the U.S. Customary system, the consistent units are G [psi], T [$lb \cdot$ in] , and L [in.], and J [$in.^4$]; in the SI system, the consistent units are G [Pa], T [N · m], L [m], and J [m^4].
- 2.The unit of θ in Eqs. (3.4) is radians, regardless of which system of unit is used in the computation.
- 3.Represent torques as vectors using the right-hand rule, as illustrated in Fig. 3.5. The same sign convention applies to the angle of twist θ .

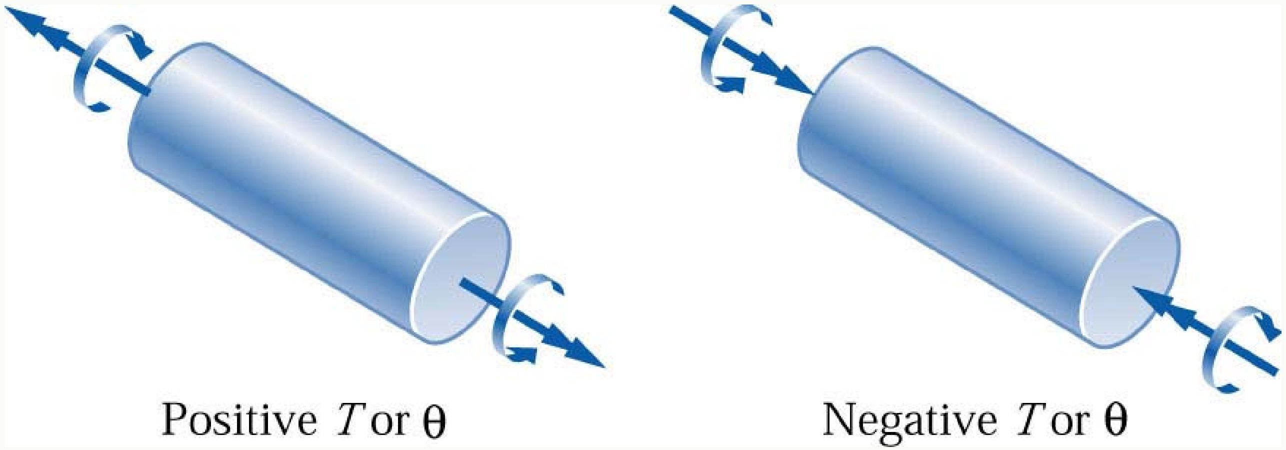


Figure 3.5 Sign Conventions for Torque *T* and angle of twist *T*.

d. *Torsion formulas*

- $G\,(d\theta/dx) = T/J$, which substitution into Eq. (3.2), $\tau = G\gamma = G\dfrac{d\theta}{dx}\rho$ gives the shear stress τ acting at the distance ρ from the center of the shaft, ***Torsion formulas*** :

$$\tau = \frac{T\rho}{J} \qquad (3.5a)$$

The **maximum** shear stress τ_{max} is found by replacing ρ by the radius r of the shaft:

$$\tau_{max} = \frac{Tr}{J} \qquad (3.5b)$$

- Because Hook´s law was used in the derivation of Eqs. (3.2)-(3.5), these formulas are valid if the shear stresses do not exceed the proportional limit of the material shear. Furthermore, these formulas are applicable only to **circular shafts**, either solid or hollow.

- The expressions for the polar moments of circular areas are :

Solid shaft :
$$\tau_{\max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} \qquad (3.5c)$$

Hollow shaft :
$$\tau_{\max} = \frac{2TR}{\pi\left(R^4 - r^4\right)} = \frac{16TD}{\pi\left(D^4 - d^4\right)} \qquad (3.5d)$$

Equations (3.5c) and (3.5d) are called the ***torsion formulas***.

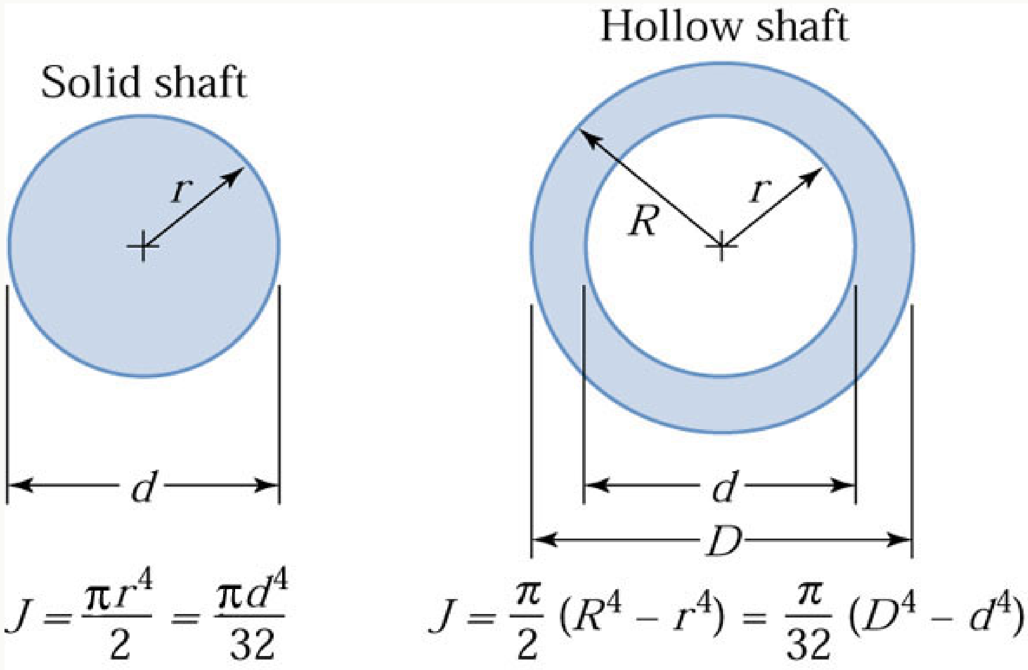


Figure 3.6 Polar moments of inertia of circular areas.

e. *Power transmission*

- Shafts are used to transmit power. The power ζ transmitted by a torque T rotating at the angular speed ω is given by $\zeta = T\omega$, where ω is measured in radians per unit time.
- If the shaft is rotating with a frequency of f revolutions per unit time, then $\omega = 2\pi f$, which gives $\zeta = T(2\pi f)$. Therefore, the torque can be expressed as

$$T = \frac{\zeta}{2\pi f} \qquad (3.6a)$$

- In SI units, ζ in usually measured in watts (1.0 W=1.0 N • m/s) and f in hertz (1.0 Hz = 1.0 rev/s); Eq. (3.6a) then determines the torque T in N • m.
- In U.S. Customary units with ζ in *lb* • in./s and f in hertz, Eq.(3.6a) calculates the torque T in *lb* • in.

- Because power in U.S. Customary units is often expressed in horsepower (1.0 hp = 550 *l*b • ft/s = 396×10³ *l*b • in./min), a convenient form of Eq.(3.6a) is

$$T(lb \cdot in) = \frac{\zeta(hp)}{2\pi f(rev/\min)} \times \frac{396 \times 10^{3}(lb \cdot in./\min)}{1.0(hp)}$$

which simplifies to

$$T(lb \cdot in) = 63.0 \times 10^{3} \frac{\zeta(hp)}{f(rev/\min)} \quad (3.6b)$$

f. *Statically indeterminate problems*

- Draw the required **free-body diagrams** and write the equations of **equilibrium**.
- Derive the **compatibility** equations from the restrictions imposed on the angles of twist.
- Use the **torque- twist relationships** in Eqs.(3.4) to express the angles of twist in the compatibility equations in terms of the torques.
- **Solve the equations** of equilibrium and compatibility for the torques.

Sample Problem 3.1

A solid steel shaft in a rolling mill transmits 20 kW of power at 2 Hz. Determine the smallest safe diameter of the shaft if the shear stress τ_w is not to exceed 40 MPa and the angle of twist θ is limited to 6° in a length of 3 m. Use G = 83 GPa.

Solution

Applying Eq. (3.6a) to determine the torque:

$$T = \frac{P}{2\pi f} = \frac{20 \times 10^3}{2\pi(2)} = 1591.5 N \cdot m$$

To satisfy the strength condition, we apply the torsion formula, Eq. (3.5c):

$$\tau_{max} = \frac{Tr}{J} \qquad \tau_{max} = \frac{16T}{\pi d^3} \qquad 4 \times 10^6 = \frac{16(1591.5)}{\pi d^3}$$

Which yields $d = 58.7 \times 10^{-3}$ m = 58.7 mm.

Apply the torque-twist relationship, Eq. (3.4b), to determine the diameter necessary to satisfy the requirement of rigidity (remembering to convert θ from degrees to radians):

$$\theta = \frac{TL}{GJ} \qquad 6\left(\frac{\pi}{180}\right) = \frac{1591.5(3)}{(83\times10^{9})(\pi d^{4}/32)}$$

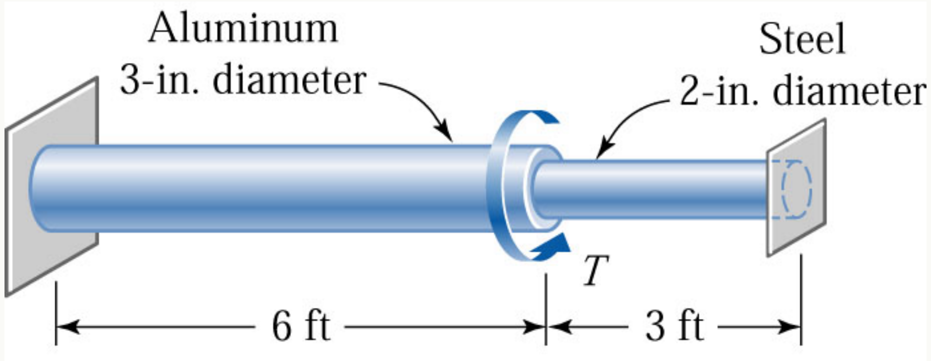
From which we obtain $d = 48.6\times10^{-3}$ m = 48.6 mm.

To satisfy both strength and rigidity requirements, we must choose the larger diameter-namely,

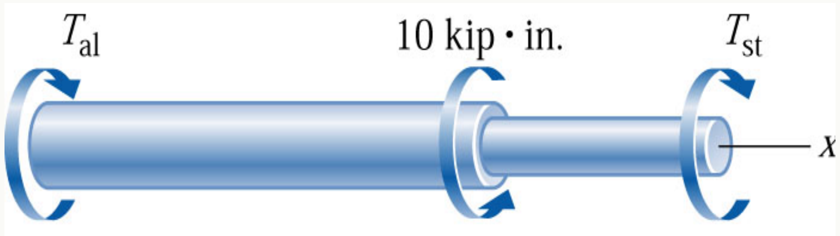
d = 58.7 mm. *Answer*

Sample problem 3.2

The shaft in Fig. (a) consists of a 3-in. -diameter aluminum segment that is rigidly joined to a 2-in. -diameter steel segment. The ends of the shaft are attached to rigid supports, Calculate the maximum shear stress developed in each segment when the torque $T = 10$ kip in. is applied. Use $G = 4\times10^6$ psi for aluminum and $G = 12\times10^6$ psi for steel.



(a)



(b) FBD

Solution

Equilibrium $\quad \Sigma M_x = 0\,, \quad (10\times10^3) - T_{st} - T_{al} = 0 \qquad (a)$

This problem is statically indeterminate.

Compatibility the two segments must have the same angle of twist; that is, $\theta_{st} = \theta_{al}$ From Eq. (3.4b), this condition between.

$$\left(\frac{TL}{GJ}\right)_{st} = \left(\frac{TL}{GJ}\right)_{al} \qquad \frac{T_{st}(3\times 12)}{\left(12\times 10^{6}\right)\frac{\pi}{32}(2)^{4}} = \frac{T_{al}(6\times 12)}{\left(4\times 10^{6}\right)\frac{\pi}{32}(3)^{4}}$$

from which

$$T_{\text{st}} = 1.1852\ T_{\text{al}} \qquad \text{(b)}$$

Solving Eqs. (a) and (b), we obtain

$$T_{\text{al}} = 4576\ lb \bullet \text{ in.} \qquad T_{\text{st}} = 5424\ lb \bullet \text{ in.}$$

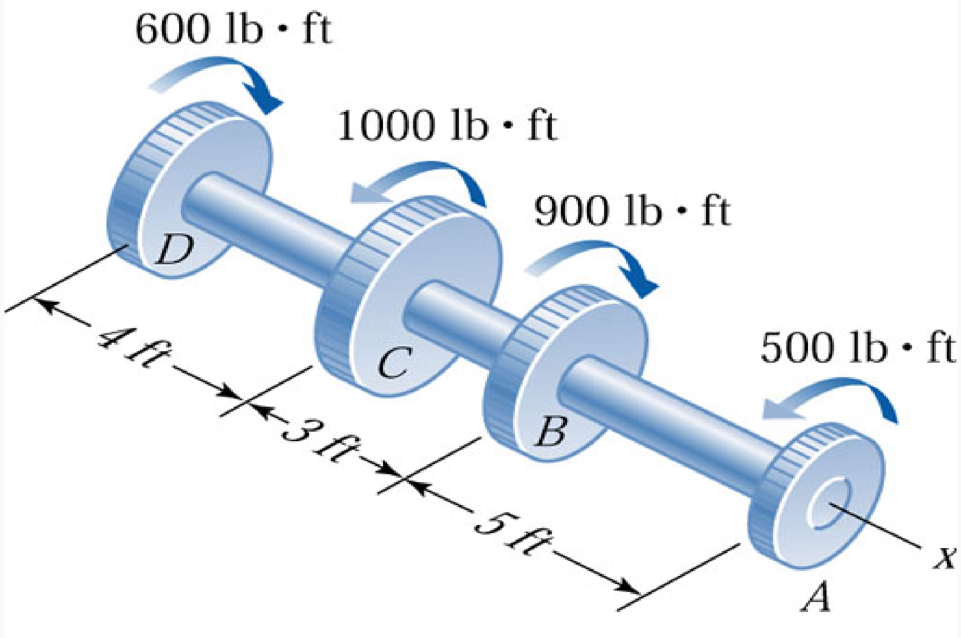
the maximum shear stresses are

$$(\tau_{\max})_{al} = \left(\frac{16T}{\pi d^{3}}\right)_{al} = \frac{16(4576)}{\pi(3)^{3}} = 863\,psi \qquad \textit{Answer}$$

$$(\tau_{\max})_{st} = \left(\frac{16T}{\pi d^{3}}\right)_{st} = \frac{16(5424)}{\pi(2)^{3}} = 3450\,psi \qquad \textit{Answer}$$

***Sample problem* 3.3**

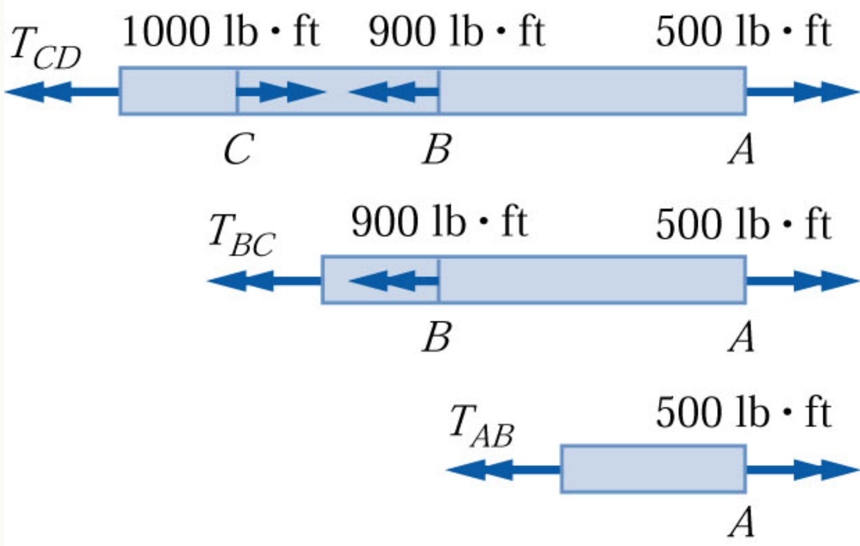
The four rigid gears, loaded as shown in Fig. (a), are attached to a 2-in.-diameter steel shaft. Compute the angle θ of rotation of gear A relative to gear D. Use $G = 12 \times 10^6$ psi for the shaft.



(a)

Solution

It is convenient to represent the torques as vectors (using the right-hand rule) on the FBDs in Fig. (b).



(b) FBDs

Solution

Assume that the internal torques T_{AB}, T_{BC}, and T_{CD} are positive according to the sign convention introduced earlier (positive torque vectors point away from the cross section). Applying the equilibrium condition $\sum M_x = 0$ to each FBD, we obtain

$$500 - 900 + 1000 - T_{CD} = 0$$

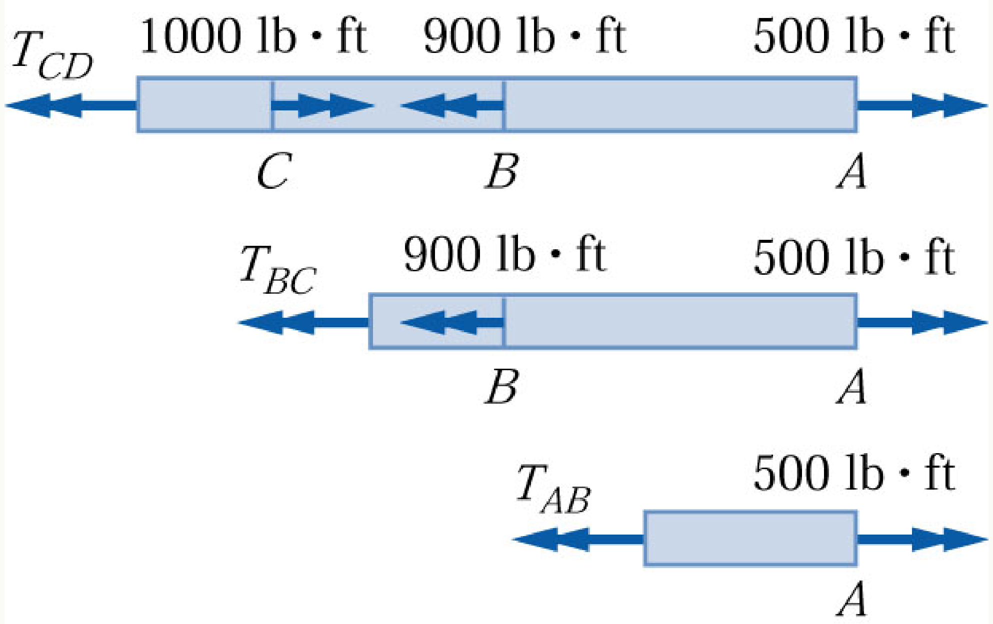
$$500 - 900 - T_{BC} = 0$$

$$500 - T_{AB} = 0$$

$$T_{AB} = 500\ lb \cdot \text{ft},$$

$$T_{BC} = -400\ lb \cdot \text{ft}$$

$$T_{CD} = 600\ lb \cdot \text{ft}$$



(b) FBDs

The minus sign indicates that the sense of T_{BC} is opposite to that shown on the FBD.*A* is gear *D* were fixed.

This rotation is obtained by summing the angles of twist of the three segments:

$$\theta_{A/D} = \theta_{A/B} + \theta_{B/C} + \theta_{C/D}$$

Using Eq.(3.4b), we obtain (converting the lengths to inches and torques to pound-inches)

$$\theta_{A/D} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}$$

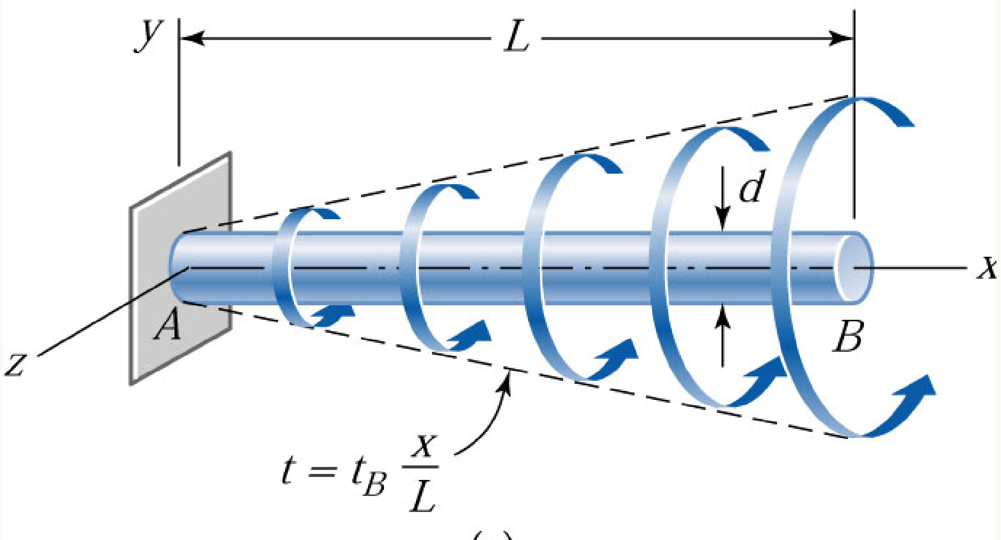
$$= \frac{(500\times12)(5\times12) - (400\times12)(3\times12) + (600\times12)(4\times12)}{\left[\pi(2)^4/32\right](12\times10)^6}$$

$$= 0.02827 \text{ rad} = 1.620^\circ \qquad \textit{Answer}$$

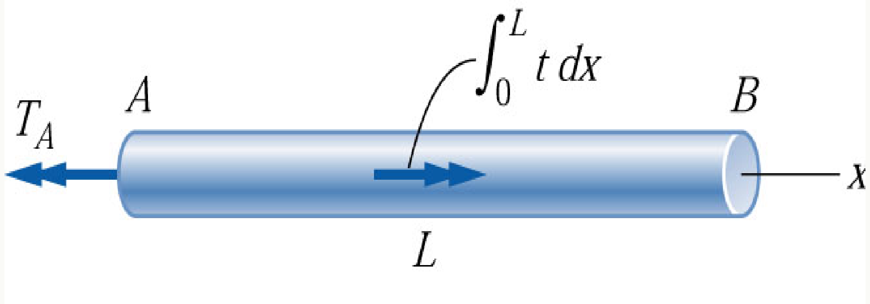
The positive result indicates that the rotation vector of *A* relative to *D* is in the positive *x*-direction: that is, θ_{AD} is directed counterclockwise when viewed from *A* toward *D*.

Sample Problem 3.4

Figure (a) shows a steel shaft of length $L = 1.5$ m and diameter $d = 25$ mm that carries a distributed toque of intensity (torque per unit length) $t = t_B(x/L)$, where $t_B = 200$ N· m/m. Determine (1) the maximum shear stress in the shaft; and (2) the angle of twist. Use $G = 80$ GPa for steel.



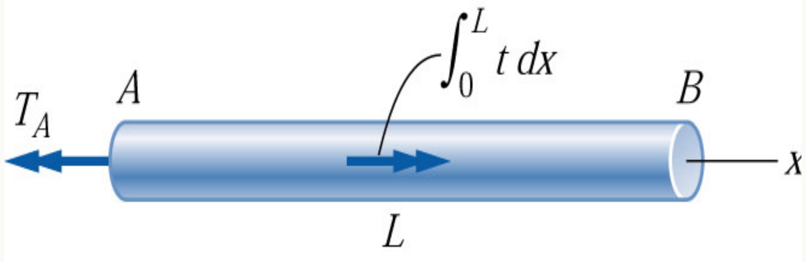
(a)



(b) FBD

Figure (a) and (b) FBD

Solution



(b) FBD

Part 1

Figure (b) shows the FBD of the shaft. The total torque applied to the shaft is $\int_0^t tdx$. The maximum torque in the shaft is T_A, which occurs at the fixed support. From the FBD we get

$$\sum M_X = 0 \qquad \int_0^L tdx - T_A = 0$$

$$T_A = \int_0^L tdx = \int_0^L t_B \frac{x}{L} dx = \frac{t_B L}{2} = \frac{1}{2}(200)(1.5) = 150 N \cdot m$$

From Eq. (3.5c), the maximum stress in the shaft is

$$\tau_{\max} = \frac{16T_A}{\pi d^3} = \frac{16(150)}{\pi(0.025)^3} = 48.9 \times 10^6 Pa = 48.9 MPa \quad Answer$$

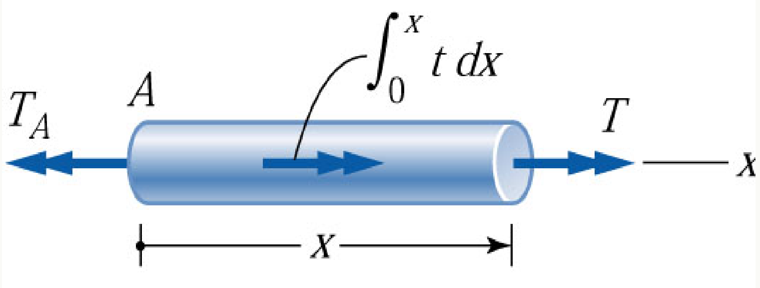
Part 2

The torque T acting on a cross section located at the distance x from the fixed end can be found from the FBD in Fig. (c):

$$\sum M_X = 0 \qquad T + \int_0^x t\,dx - T_A = 0$$

$$T = T_A - \int_0^x t\,dx = \frac{t_B L}{2} - \int_0^x t_B \frac{x}{L} dx$$

$$= \frac{t_B}{2L}\left(L^2 - x^2\right)$$



(c) FBD

From Eq. (3.4a), the angle θ of twist of the shaft is

$$\theta = \int_0^L \frac{T}{GJ} dx = \frac{t_B}{2LGJ}\int_0^L \left(L^2 - x^2\right)dx = \frac{t_B L^2}{3GJ}$$

$$= \frac{200(1.5)^2}{3\left(80\times10^9\right)(\pi/32)(0.025)^4} = 0.0489 rad = 2.8^\circ \qquad Answer$$