Review in Linear Systems

Mathematical Background, Examples & Engineering Applications

Solving Ax=b

Gaussian Elimination.

This method is good for many problems, but has drawbacks:

- Round-off error can accumulate, especially if diagonal elements are small
- For "**sparse**" matrices, algorithm is inefficient.

`sparse.pdf`

Απαλοιφή Gauss

(πηγή: Πανεπιστήμιο Αιγαίου, Τμήμα Μαθηματικών, Χρ.Νικολόπουλος, 2008)

Η βασική ιδέα της μεθόδου είναι να μετατρέψουμε ένα σύστημα της μορφής $Ax = b$, όπου $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, και με αγνώστους το διάνυσμα $x \in \mathbb{R}^n$, σε ένα ισοδύναμο σύστημα της μορφής $A'x = b'$ όπου ο πίνακας A' είναι άνω τριγωνικός, δηλαδή έχει στοιχεία $a'_{i,j} = 0$ για $i > j$. Αυτή η διαδικασία ονομάζεται άνω τριγωνοποίηση. Πιο συγκεκριμένα πολλαπλασιάζουμε τα μέλη της πρώτης εξίσωσης διαδοχικά επί $-\frac{a_{21}}{a_{11}}$, $-\frac{a_{31}}{a_{11}}, \ldots, -\frac{a_{n1}}{a_{11}}$ και τις νέες εξισώσεις αντίστοιχα τις προσθέτουμε στην δεύτερη, τρίτη, ..., n-οστή εξίσωση.

Παρατήρηση Οδήγηση είναι η διαδικασία κατά την οποία βρίσκουμε τον μεγαλύτερο σε απόλυτη τιμή συντελεστή των αγνώστων (οδηγό στοιχείο) και τον φέρνουμε στη θέση του συντελεστή a_{11} αλλάζοντας τη σειρά των γραμμών και των στηλών του συστήματος. Αν η οδήγηση γίνεται σε κάθε βήμα της άνω τριγωνοποίηση τότε λέγεται πλήρης οδήγηση. Με τη οδήγηση εξασφαλίζεται η ευστάθεια της διαδικασίας της άνω τριγωνοποίησης.

Μερική οδήγηση κατά στήλη είναι η διαδικασία κατα την οποία σε κάθε βήμα της τριγωνοποίησης, πριν την απαλοιφή του αγνώστου x_i ανταλλάσουμε την εξίσωση i με την εξίσωση $j \geq i$ που έχει μεγαλύτερο συντελεστή κατά απόλυτη τιμή. Μερική οδήγηση κατά γραμμή είναι η διαδικασία κατά την οποία σε κάθε βήμα της τριγωνοποίησης, πριν την απαλοιφή του αγνώστου x_i ανταλλάσουμε τον άγνωστο x_i με τον άγνωστο x_j, $j \geq i$ που έχει μεγαλύτερο συντελεστή κατα απόλυτη τιμή.

Γενική Επαναληπτική Μέθοδος

Για ένα πίνακα $A \in \mathbb{R}^{n\times n}$ με ιδιοτιμές $\lambda_1, \lambda_2, \ldots, \lambda_n$, πραγματικές ή μιγαδικές, ο αριθμός

$$\rho(A) = \max_{1\le i\le n} |\lambda_i|,$$

ονομάζεται φασματική ακτίνα του πίνακα A.

Για την επίλυση ενός γραμμικού συστήματος $Ax = b$ με $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, όπου A είναι αντιστρέψιμος πίνακας και το σύστημα έχει μοναδική λύση γράφουμε αρχικά τον πίνακα A σαν διαφορά δύο πινάκων P και Q, $A = P - Q$, όπου ο Q είναι πίνακας με μηδενική ορίζουσα.

Έχουμε $A = Q - P$ και άρα $(Q - P)x = b$ ή $x = Q^{-1}Px + Q^{-1}b$ ή $x = Cx + d$, για $C = Q^{-1}P$, και $d = Q^{-1}b$. Ορίζουμε την επαναληπτική μέθοδο ώς εξής:
$x_k = Cx_{k-1} + d_k$ με αρχικό διάνυσμα x_0 δεδομένο.

Η ακολουθία $\{x_k\}_{k=0}^{\infty}$ που κατασκευάζεται με βάση την επαναληπτική μέθοδο $x_k = Cx_{k-1} + d_k$ συγκλίνει στη λύση x του συστήματος $Ax = b$ αν και μονο αν $\lim_{k\to\infty} C^k = 0$.
Επιπλέον εναλλακτικά έχουμε ότι η ακολουθία $\{x_k\}_{k=0}^{\infty}$ συγκλίνει στη λύση x αν και μόνο αν $\rho(C) < 1$.

Παράδειγμα

Solve the following system of simultaneous linear equations by carrying out two iterations of the Gauss-Seidel method using the initial vector $(0.3, 0.3, 0.25)^{\mathsf{T}}$:

$$\begin{pmatrix} 20 & -8 & 0 \\ -4 & 20 & 4 \\ 0 & -4 & 20 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix}.$$

Show that the Gauss-Seidel method will converge from any initial vector $\boldsymbol{x}_0$ for this set of equations.

The Gauss-Seidel method carries out iterations of the form

$$\boldsymbol{x}_{n+1} = \boldsymbol{T}\boldsymbol{x}_n + \boldsymbol{c},$$

with

$$\boldsymbol{T} = -(\boldsymbol{L}+\boldsymbol{D})^{-1}\boldsymbol{U}, \quad \boldsymbol{c} = (\boldsymbol{L}+\boldsymbol{D})^{-1}\boldsymbol{b}.$$

and $\boldsymbol{A} = \boldsymbol{L}+\boldsymbol{D}+\boldsymbol{U}$ with $\boldsymbol{L}$,$\boldsymbol{D}$ and $\boldsymbol{U}$ denoting the lower triangular, diagonal, and upper triangular parts, respectively.

the Gauss-Seidel iteration scheme for this system of equations is given as

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{1}{20}\left(6 + 8x_2^{(k)}\right) \\
x_2^{(k+1)} &= \frac{1}{20}\left(6 + 4x_1^{(k+1)} - 4x_3^{(k)}\right) \\
x_3^{(k+1)} &= \frac{1}{20}\left(5 + 4x_2^{(k+1)}\right).
\end{aligned}
$$

Starting with the initial value $\boldsymbol{x}_0 = (0.3, 0.3, 0.25)^{\mathsf{T}}$, we get

$$
\begin{aligned}
x_1^{(1)} &= \frac{1}{20}(6 + 8 \cdot 0.3) = 0.42 \\
x_2^{(1)} &= \frac{1}{20}(6 + 4 \cdot 0.42 - 4 \cdot 0.25) = 0.3340 \\
x_3^{(1)} &= \frac{1}{20}(5 + 4 \cdot 0.3340) = 0.3168.
\end{aligned}
$$

and

$$
\begin{aligned}
x_1^{(2)} &= \frac{1}{20}(6 + 8 \cdot 0.42) = 0.4336 \\
x_2^{(2)} &= \frac{1}{20}(6 + 4 \cdot 0.4336 - 4 \cdot 0.3168) = 0.3234 \\
x_3^{(2)} &= \frac{1}{20}(5 + 4 \cdot 0.3234) = 0.3147.
\end{aligned}
$$

The method converges if and only if $\rho(\boldsymbol{T}) < 1$. To verify this, we compute the eiganvalues of $\boldsymbol{T}$ as the roots of the polynomial

$$\det(\lambda(\boldsymbol{L} + \boldsymbol{D}) + \boldsymbol{U}) = 0.$$

Written out, this determinant looks like

$$\det \begin{pmatrix} 20\lambda & -8 & 0 \\ -4\lambda & 20\lambda & 4 \\ 0 & -4\lambda & 20\lambda \end{pmatrix} = 20\lambda(400\lambda^2 + 16\lambda) + 8(-80\lambda^2)$$

$$= 8000\lambda^3 - 320\lambda^2 = 0.$$

Two of the roots are 0, while the third root is

$$\lambda = \frac{320}{8000} = 0.04.$$

This is also the value of the spectral radius, which is clearly < 1, and therefore the method converges.

Jacobi Example

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix} \qquad x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1, i \neq j}^{n} a_{ij} x_j^{(k)})$$

$$x^{(k+1)} = \frac{1}{4}(11 - \frac{2}{4} y^{(k)} - \frac{1}{4} z^k)$$

$$y^{(k+1)} = \frac{1}{2}(3 - \frac{-1}{2} x^{(k)} - \frac{0}{2} z^k)$$

$$z^{(k+1)} = \frac{1}{4}(16 - \frac{2}{4} x^{(k)} - \frac{1}{4} y^k)$$

Jacobi Example

Start with (x,y,z) =(1,1,1). After one iteration we have

$X^1 = (11/4) - (1/2)Y^0 - (1/4)Z^0 = 2$

$Y^1 = (3/2) + (1/2)X^0 = 2$

$Z^1 = 4 - (1/2)X^0 - (1/4)Y^0 = 13/4$

We need a stopping condition. We will stop when **$\max(x^k - x^{k-1}, y^k - y^{k-1}, z^k - z^{k-1}) < 0.1$**

- Second iteration:

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2)X^1 - (1/4)Y^1 = 5/2$

Jacobi Example

Converging Process:

$$[1,1,1],\ \left[2,2,\frac{13}{4}\right],\ \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right],\ \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right],\ \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right].$$ Stop the iteration when

$$\max\left|X^5 - X^4, Y^5 - Y^4, Z^5 - Z^4\right| \leq 0.1$$

- **Final solution [1.014, 2.02, 2.996]**
- **Exact solution [1, 2, 3]**

Gauss-Seidel Example

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z,\ Y = \frac{3}{2} + \frac{1}{2}X,\ Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$

Immediate substitution

$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: $[1,1,1],\ \left[2, \frac{5}{2}, \frac{19}{8}\right],\ \left[\frac{29}{32}, \frac{125}{64}, \frac{783}{256}\right],\ \left[\frac{1033}{1024}, \frac{4095}{2048}, \frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster

Jacobi Method

Example

Consider 4x4 case

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$$

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 - 8x_4 &= 15 \end{aligned}$$

$$\begin{aligned} x_1 &= (x_2 - 2x_3 + 6)/10 \\ x_2 &= (x_1 + x_3 - 3x_4 + 25)/11 \\ x_3 &= (-2x_1 + x_2 + x_4 - 11)/10 \\ x_4 &= (-3x_2 + x_3 + 15)/(-8) \end{aligned}$$

given $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} x_1^{(1)} &= (x_2^{(0)} - 2x_3^{(0)} + 6)/10 \\ x_2^{(1)} &= (x_1^{(0)} + x_3^{(0)} - 3x_4^{(0)} + 25)/11 \\ x_3^{(1)} &= (-2x_1^{(0)} + x_2^{(0)} + x_4^{(0)} - 11)/10 \\ x_4^{(1)} &= (-3x_2^{(0)} + x_3^{(0)} + 15)/(-8) \end{aligned}$$

$$x^{(1)} = \begin{bmatrix} 0.6000 \\ 2.2727 \\ -1.1000 \\ 1.8750 \end{bmatrix}$$

Jacobi Method

$$x_1^{(k+1)} = (\quad x_2^{(k)} - 2x_3^{(k)} \quad + 6)/10$$
$$x_2^{(k+1)} = (x_1^{(k)} \quad + x_3^{(k)} - 3x_4^{(k)} + 25)/11$$
$$x_3^{(k+1)} = (-2x_1^{(k)} + x_2^{(k)} \quad + x_4^{(k)} - 11)/10$$
$$x_4^{(k+1)} = (\quad - 3x_2^{(k)} + x_3^{(k)} \quad + 15)/(-8)$$

	K=1	K=2	K=3	K=4	K=5
x1	0.6000	1.0473	0.9326	1.0152	0.9890
x2	2.2727	1.7159	2.0533	1.9537	2.0114
x3	-1.1000	-0.8052	-1.0493	-0.9681	-1.0103
X4	1.8750	0.8852	1.1309	0.9738	1.0214

	K=6	K=7	K=8	K=9	K=10
x1	1.0032	0.9981	1.0006	0.9997	1.0001
x2	1.9922	2.0023	1.9987	2.0004	1.9998
x3	-0.9945	-1.0020	-0.9990	-1.0004	-0.9998
X4	0.9944	1.0036	0.9989	1.0006	0.9998

$$x^* = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Gauss Seidel Method

Note that in the Jacobi iteration one does not use the most recently available information.

$$x_1^{(k+1)} = (\quad x_2^{(k)} - 2x_3^{(k)} + 6)/10$$
$$x_2^{(k+1)} = (x_1^{(k)} + x_3^{(k)} - 3x_4^{(k)} + 25)/11$$
$$x_3^{(k+1)} = (-2x_1^{(k)} + x_2^{(k)} + x_4^{(k)} - 11)/10$$
$$x_4^{(k+1)} = (\quad -3x_2^{(k)} + x_3^{(k)} + 15)/(-8)$$

$$x_1^{(k+1)} = (\quad x_2^{(k)} - 2x_3^{(k)} + 6)/10$$
$$x_2^{(k+1)} = (x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)} + 25)/11$$
$$x_3^{(k+1)} = (-2x_1^{(k+1)} + x_2^{(k+1)} + x_4^{(k)} - 11)/10$$
$$x_4^{(k+1)} = (\quad -3x_2^{(k+1)} + x_3^{(k+1)} + 15)/(-8)$$

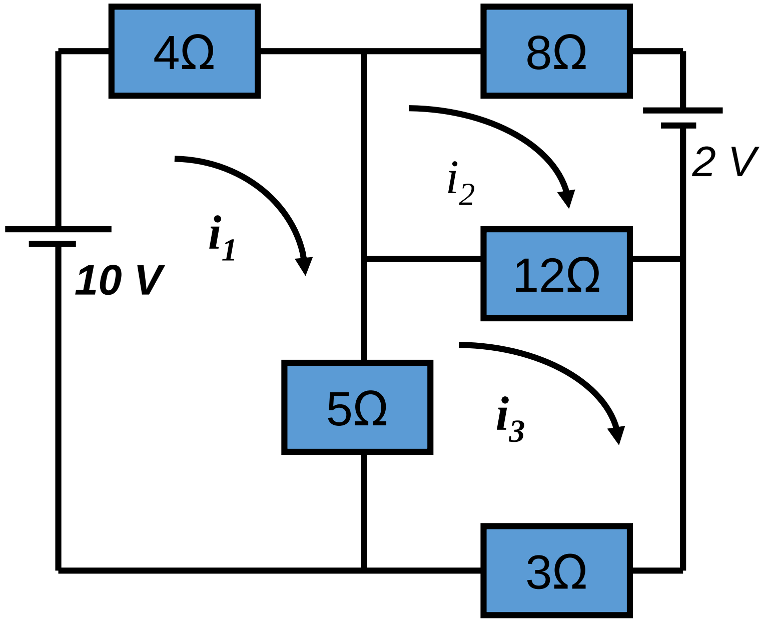
	K=1	K=2	K=3	K=4	K=5
x1	0.6000	1.0302	1.0066	1.0009	1.0001
x2	2.3273	2.0369	2.0036	2.0003	2.0000
x3	-0.9873	-1.0145	-1.0025	-1.0003	-1.0000
X4	0.8789	0.9843	0.9984	0.9998	1.0000

$$x^* = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Jacobi method :

Consider a circuit shown in figure here; currents i_1, i_2, and i_3 are given by

$$9i_1 + 0i_2 - 5i_3 = 10$$
$$0i_1 + 20i_2 - 12i_3 = -2$$
$$-5i_1 - 12i_2 + 20i_3 = 0$$



The matrix form is:

$$\begin{bmatrix} 9 & 0 & -5 \\ 0 & 20 & -12 \\ -5 & -12 & 20 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

Notice that magnitude of any diagonal element is greater than the sum of the magnitudes of other elements in that row

A matrix with this property is said to be **Diagonally dominant**.

Jacobi method :

The set of equations:

$$9i_1 + 0i_2 - 5i_3 = 10$$

$$0i_1 + 20i_2 - 12i_3 = -2$$

$$-5i_1 - 12i_2 + 20i_3 = 0$$

Let us write for i_1, i_2 and i_3 as

$$i_1 = \ 10 + 5i_3 \ / 9 = 1.1111 + 0.5556i_2 \qquad (1)$$

$$i_2 = \ -2 + 12i_3 \ / 20 = -0.1000 + 0.6000i_2 \qquad (2)$$

$$i_3 = \ 5i_1 + 12i_3 \ / 20 = 0.2500i_1 + 0.6000i_2 \qquad (3)$$

Let us make an initial guess as *$i_1 = 0.0$; $i_2 = 0.0$ and $i_3 = 0.0$*

First iteration results: *$i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.0$*

Jacobi method :

$$i_1 = (10 + 5i_3)/9 = 1.1111 + 0.5556i_2 \qquad (1)$$

$$i_2 = (-2 + 12i_3)/20 = -0.1000 + 0.6000i_2 \qquad (2)$$

$$i_3 = (5i_1 + 12i_3)/20 = 0.2500i_1 + 0.6000i_2 \qquad (3)$$

First iteration results: $i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.0$

2nd iteration results: $i_1 = 1.1111$; $i_2 = -0.1000$ and $i_3 = 0.22$

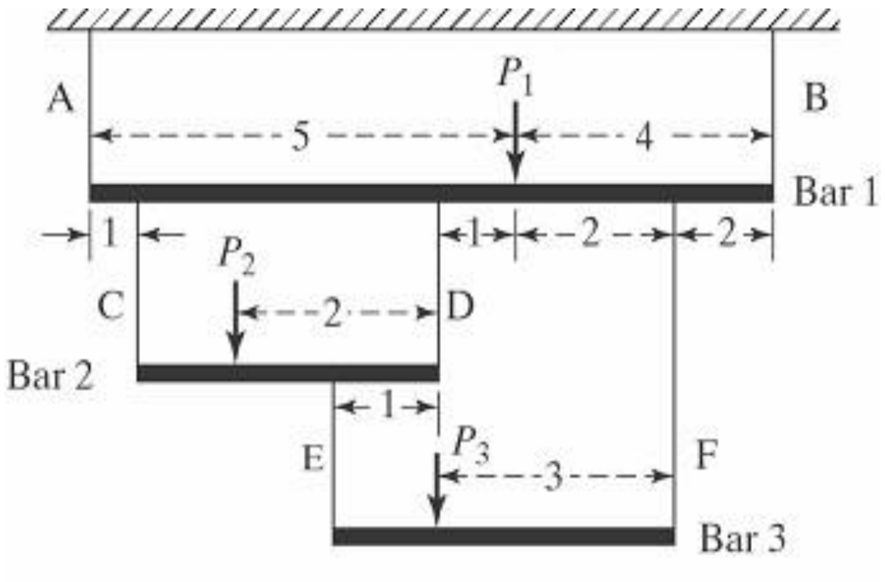
3rd iteration results: $i1 = 1.23$; $i2 = 0.03$ and $i3 = 0.22$

4th iteration results: $i1 = 1.23$; $i2 = 0.03$ and $i3 = 0.33$

5th iteration results: $i1 = 1.29$; $i2 = 0.1$ and $i3 = 0.33$

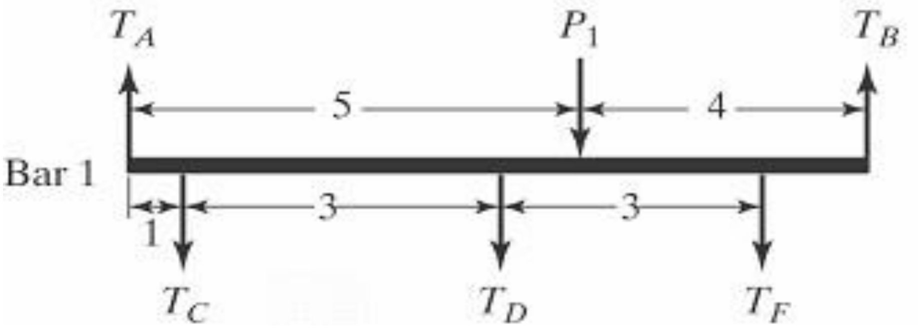
6th iteration results: $i1 = 1.29$; $i2 = 0.1$ and $i3 = 0.38$

Solution of Simultaneous Linear Algebraic Equations: Static Analysis of a Scaffolding



3 bars supported by 6 cables form a simple scaffolding. Given the positions and magnitudes for 3 loads applied to the bars, find the tension in each cable.

Governing Equations for Bar 1



Force equilibrium

$$\Sigma F_y = 0$$

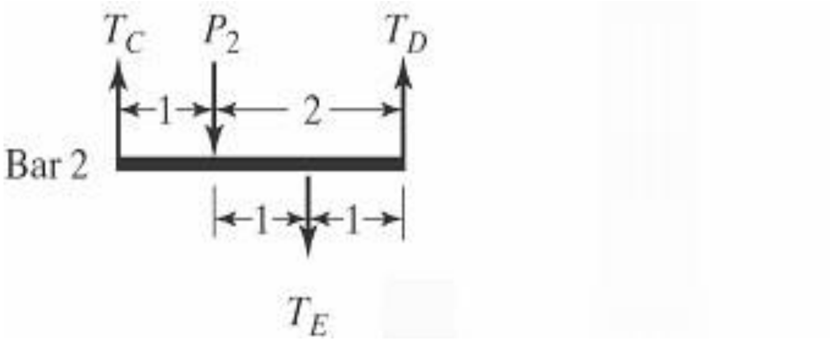
$$T_A + T_B - T_C - T_D - T_F - P_1 = 0$$

Moment equilibrium

$$\Sigma M = 0$$

$$-9T_B + T_C + 4T_D + 7T_F + 5P_1 = 0$$

Governing Equations for Bar 2



Force equilibrium

$$\Sigma F_y = 0$$

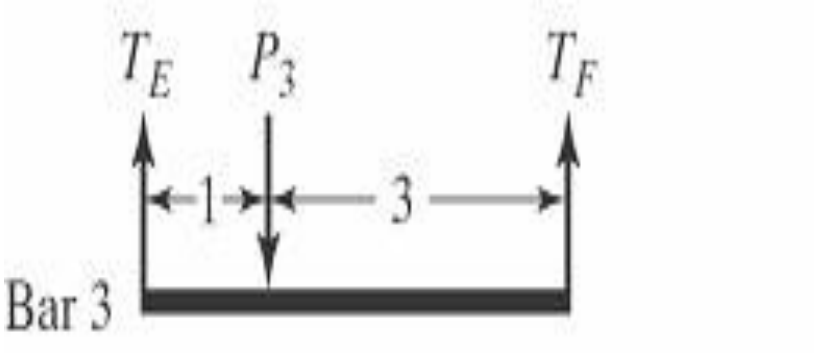
$$T_C + T_D - T_E - P_2 = 0$$

Moment equilibrium

$$\Sigma M = 0$$

$$-3T_D + 2T_E + P_2 = 0$$

Governing Equations for **Bar 3**



Force equilibrium

$$\Sigma F_y = 0$$

$$T_E + T_F - P_3 = 0$$

Moment equilibrium

$$\Sigma M = 0$$

$$-4T_F + P_3 = 0$$

Assembling Equations

At this point, we have six independent equations (two for each bar), and six unknowns (cable tensions). Reformat the six equilibrium equations to isolate the unknown tensions on the left-hand side of the equations. Make sure the tension variables are in the same order in each equation:

$$
\begin{aligned}
T_A + T_B - T_C - T_D - T_F &= P_1 \\
-9T_B + T_C + 4T_D + 7T_F &= -5P_1 \\
T_C + T_D - T_E &= P_2 \\
-3T_D + 2T_E &= -P_2 \\
T_E + T_F &= P_3 \\
-4T_F &= -P_3
\end{aligned}
$$

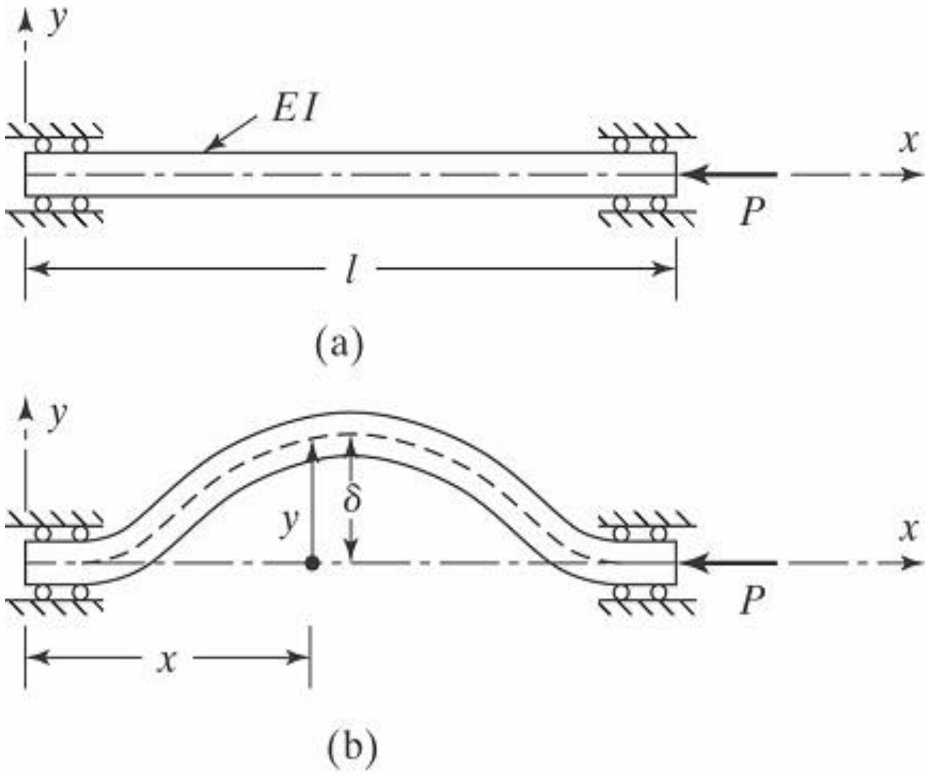
Solution of Governing Equations

If $P_1 = 2000$ lb, $P_2 = 1000$ lb, $P_3 = 500$ lb, various matrix algebra methods can solve for $T_A \cdots T_F$:

T_A = 1944.45 lb
T_C = 791.67 lb
T_E = 375 lb

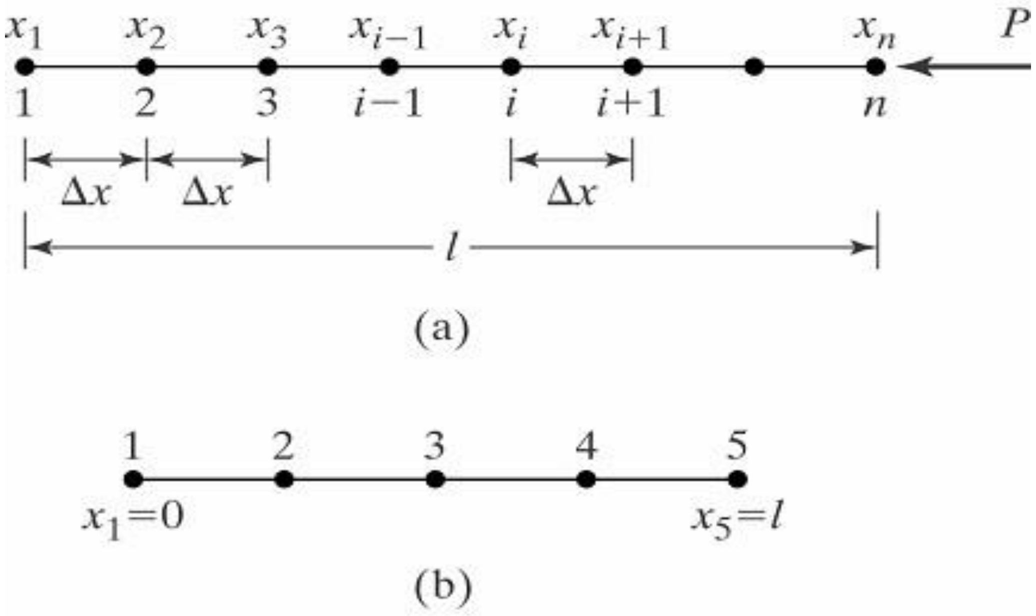
T_B = 1555.55 lb
T_D = 583.33 lb
T_F = 125 lb

Eigenvalue Problems: Critical Loads for Buckling a Column



A long column with elastic modulus *E* and cross-sectional moment of inertia *I* is subjected to an axial load *P*. If there is a small deformity in the column due to misalignment during construction or some other reason, its strength is considerably reduced. The deformity will cause the column to buckle long before a shorter column would have been crushed.

Governing Equations for Discretized Column



The continuous differential equation of deflection

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

can be discretized with the following substitution:

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

Solution of Discretized Equations

At any given point i, the governing equation evaluates to

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} + \lambda y_i = 0$$

where **$\lambda = P/(EI)$.** Dividing the column into 4 segments (a total of 5 points), evaluating the equation at points 2, 3, and 4 yields:

$$y_1 - \quad 2 - \frac{\lambda L^2}{16} \quad y_2 + y_3 = 0$$

$$y_2 - \quad 2 - \frac{\lambda L^2}{16} \quad y_3 + y_4 = 0$$

$$y_3 - \quad 2 - \frac{\lambda L^2}{16} \quad y_4 + y_5 = 0$$

Since the column is pinned on both ends, we assume that the deflections $y_1 = y_5 = 0$.