# Iteration Methods

- **Iterative methods**: produce a sequence of approximate solutions hopefully converging to the exact solution
  - **Stationary**
    - Jacobi
    - Gauss-Seidel
    - SOR (Successive Over-relaxation Method)
  - **Non Stationary**
    - GCR, CG, GMRES…..

# Introduction

- Why are we using iteration methods?

- If systems of linear equations are very large, the computational effort of direct methods is prohibitively expensive

- Iterative solutions have the characteristic that they converge to a solution from an initial guess.

- Iterative solutions are efficient for very large problems, e.g. A of order 105x105 or larger <u>particularly where A is sparse</u>.

- We need to know the types of problems for which these iterative methods are applicable and why.

- We need to understand the conditions under which these iterative methods converge and why.

- Theoretically, methods such as Gauss Jordan elimination will give us exact solutions.

- If the system is too big, we will have problems with
  - the round-off errors
  - the storage capacity of the computer.
  - the computational effort of direct methods is prohibitively expensive

How to solve it ?

- Iteration: repeating a process over and over until an approximation of the solution is reached.

- It is useful to solve certain types of problems

- When the number of unknowns is very large but the coefficient matrix is **sparse**, Gauss Elimination becomes inefficient, and sometimes inapplicable if the methods are preferred.

# Iterative Solution Methods

- Starts with an initial approximation for the solution vector $(x^0)$

- At each iteration updates the x vector by using the sytem $Ax=b$

- During the iterations $A$, matrix is not changed so sparcity is preserved

- Each iteration involves a matrix-vector product

- If $A$ is sparse this product is efficiently done

# Iterative solution procedure

- Write the system $Ax=b$ in an equivalent form

  $x=Ex+f$ (like $x=g(x)$ for fixed-point iteration)

- Starting with $x^0$, generate a sequence of approximations $\{x^k\}$ iteratively by

$$x^{k+1}=Ex^k+f$$

- Representation of E and f depends on the type of the method used

- But for every method E and f are obtained from $A$ and $b$, but in a different way

# Convergence

- As k INCREASES, the sequence $\{x^k\}$ converges to the solution vector under some conditions on E matrix

- This imposes different conditions on $A$ matrix for different methods

- For the same $A$ matrix, one method may converge while the other may diverge

- Therefore for each method the relation between $A$ and E should be found to decide on the convergence

- For systems that have coefficient matrices with the appropriate structure – especially large, sparse systems (many coefficients whose value is zero) – iterative techniques may be preferable

$$
\begin{bmatrix}
4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{bmatrix}
\cdot
\begin{bmatrix}
u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16}
\end{bmatrix}
=
\begin{bmatrix}
0.08 \\ 0.16 \\ 0.36 \\ 1.64 \\ 0.16 \\ 0.0 \\ 0.0 \\ 1.0 \\ 0.36 \\ 0 \\ 0 \\ 1.0 \\ 1.64 \\ 1.0 \\ 1.0 \\ 2.0
\end{bmatrix}
$$

- Additional advantages of iterative methods include
  - (1) programming is simple, and
  - (2) it is easily applicable when coefficients are nonlinear.

- Although there are many versions of iterative schemes, we introduce three iterative methods:

  - **Jacobi iterative,**
  - **Gauss-Seidel, and**
  - **Successive-over-relaxation (SOR)**
    - SOR is a method used to accelerate the convergence
    - Gauss-Seidel Iteration is a special case of SOR method

# Advantages / Disadvantages

Iterative methods can be applied to system as many as 100,000 variables. Examples of these large systems arise in the solution of partial differential equations.

Also the amount of storage, as stated earlier, is far less than directs methods. In our example, we have :

- 100,000 * 100,000 variables for direct
- 3 * 100,000 – 2 variables for iteration

**Iteration cannot be applied to every system**

# Applications

(a) Electric network consisting of resistors

(b) Heat-conduction problems

(c) Particule diffusion

(d) Certain stress-strain problems

(e) Fluid, magnetic, or electric potential

# Convergence Restrictions

- There are two conditions for the iterative method to converge.
    - Necessary that 1 coefficient in each equation is dominate.
    - The sufficient condition is that the diagonal is dominate.

$$\left| a_{ii} \right| > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| a_{ij} \right|$$

That is, the absolute value of the diagonal coefficient in each of the equations must be larger than the sum of the absolute values of the other coefficients in the equation. Such systems are said to be **diagonally dominant**. This criterion is sufficient but not necessary for convergence.