ΑΡΙΘΜΗΤΙΚΗ ΑΝΑΛΥΣΗ

Αριθμητική Επίλυση Συστημάτων Γραμμικών Εξισώσεων

- Gaussian elimination
- Gauss - Jordan

Direct solution Methods

- **Gaussian Elimination**
  - Matrix A is transformed into an upper triangular matrix (all elements below diagonal 0)
  - Back substitution is used to solve the upper-triangular system

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

Back substitution ↑

Gauss Elimination
(μέθοδος απαλοιφής Gauss)

1. Gauss Elimination
2. Gauss Elimination Pitfalls
3. Gauss Elimination with Partial Pivoting
4. Determinant of a Square Matrix Using Gauss Elimination

1. Gaussian Elimination

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

The goal of forward elimination is to transform the coefficient matrix into **an upper triangular matrix**

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Forward Elimination

A set of n equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

(n-1) steps of forward elimination

Forward Elimination

Step 1

For Equation 2, divide Equation 1 by a_{11} and multiply by a_{21}

$$\left[\frac{a_{21}}{a_{11}}\right](a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1)$$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Forward Elimination

Subtract the result from Equation 2.

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$- \quad a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $$a_{22}^{'}x_2 + \ldots + a_{2n}^{'}x_n = b_2^{'}$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b'_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n$$

End of Step 1

Forward Elimination

Step 2

Repeat the same procedure for the 3rd term of Equation 3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n$$

End of Step 2

Forward Elimination

At the end of (n-1) Forward Elimination steps, the system of equations will look like

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \dots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

End of Step (n-1)

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & 0 & a''_{33} & \cdots & a''_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution

Solve each equation starting from the last equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \ldots - a_{i,n}^{(i-1)} x_n}{a_{ii}^{(i-1)}} \text{ for } i = n-1, \ldots, 1$$

$$x_i = \frac{b_i^{\,i-1} - \sum_{j=i+1}^{n} a_{ij}^{\,i-1}\, x_j}{a_{ii}^{\,i-1}} \text{ for } i = n-1, \ldots, 1$$

Computational Complexity

- Forward Elimination

For i = 1 to n-1 {   // for each equation

  For j = i+1 to n {   // for each target equation below the current

$$M_{ji} = \frac{A_{ji}}{A_{ii}}, A_{ji} = 0$$

$$\sum_{i=1}^{n-1}(n-i) = \frac{n^2}{2} \quad \text{divisions}$$

    For k = i+1 to n { // for each element beyond pivot column

$$A_{jk} \leftarrow A_{jk} - M_{ji} A_{ik}$$

$$\sum_{i=1}^{n-1}(n-i)^2 \approx \frac{2}{3}n^3$$

multiply-add's

    }
  }
}

$O(n^3)$

Computational Complexity

- Backward Substitution

```
For i = n-1 to 1 {          // for each equation
    For j = n  to i+1 {     // for each known variable
        sum = sum – A_ij * x_j
    }
}
```

$$\sum_{i=1}^{n-1}(n-i)=\frac{n^2}{2}\ \text{multiply-add's}$$

$O(n^2)$

Example 1

The upward velocity of a rocket is given at three different times

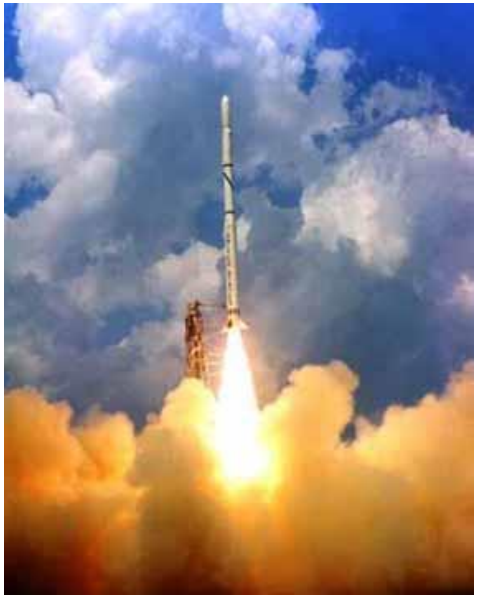

Table 1 Velocity vs. time data.

Time, $t\,(\mathrm{s})$	**Velocity,** $v\,(\mathrm{m/s})$
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 , \qquad 5 \le t \le 12.$$

Find the velocity at t=6 seconds .

Example 1 Cont.

Assume

$$v(t) = a_1 t^2 + a_2 t + a_3, \quad 5 \le t \le 12.$$

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using data from Table 1, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 1 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$(n\text{-}1)=(3\text{-}1)=2$

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 & \vdots & 273.408 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{rrrcr} & 64 & 8 & 1 & \vdots & 177.2 \\ - & 64 & 12.8 & 2.56 & \vdots & 273.408 \\ \hline & 0 & -4.8 & -1.56 & \vdots & -96.208 \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 & \vdots & 615.168 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{rrrcr} & 144 & 12 & 1 & \vdots & 279.2 \\ - & 144 & 28.8 & 5.76 & \vdots & 615.168 \\ \hline & 0 & -16.8 & -4.76 & \vdots & -335.968 \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & -16.8 & -4.76 & \vdots & -335.968 \end{bmatrix}$$

Divide Equation 2 by -4.8

and multiply it by -16.8,

$$\frac{-16.8}{-4.8} = 3.5$$

$$\begin{bmatrix} 0 & -4.8 & -1.56 & \vdots & -96.208 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 & \vdots & -336.728 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{rrrrcr} & 0 & -16.8 & -4.76 & \vdots & 335.968 \\ - & 0 & -16.8 & -5.46 & \vdots & -336.728 \\ \hline & 0 & 0 & 0.7 & \vdots & 0.76 \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.208 \\ 0 & 0 & 0.7 & \vdots & 0.76 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 0 & -4.8 & -1.56 & \vdots & -96.2 \\ 0 & 0 & 0.7 & \vdots & 0.7 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_3

$$0.7a_3 = 0.76$$

$$a_3 = \frac{0.76}{0.7}$$

$$a_3 = 1.08571$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.208 \\ 0.76 \end{bmatrix}$$

Solving for a_2

$$-4.8a_2 - 1.56a_3 = -96.208$$

$$a_2 = \frac{-96.208 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.208 + 1.56 \times 1.08571}{-4.8}$$

$$a_2 = 19.6905$$

Back Substitution (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.2 \\ 0.76 \end{bmatrix}$$

Solving for a_1

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$= \frac{106.8 - 5 \times 19.6905 - 1.08571}{25}$$

$$= 0.290472$$

Gaussian Elimination Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

Example 1 Cont.

Solution

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.290472 \\ 19.6905 \\ 1.08571 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$\begin{aligned} v(t) &= a_1 t^2 + a_2 t + a_3 \\ &= 0.290472 t^2 + 19.6905 t + 1.08571, \quad 5 \le t \le 12 \end{aligned}$$

$$\begin{aligned} v(6) &= 0.290472(6)^2 + 19.6905(6) + 1.08571 \\ &= 129.686 \text{ m/s.} \end{aligned}$$

2. Gauss Elimination Pitfalls

Pitfall#1. Division by zero

$$10x_2 - 7x_3 = 3$$
$$6x_1 + 2x_2 + 3x_3 = 11$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 0 & 10 & -7 \\ 6 & 2 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \\ 9 \end{bmatrix}$$

Is division by zero an issue here?

$$12x_1 + 10x_2 - 7x_3 = 15$$
$$6x_1 + 5x_2 + 3x_3 = 14$$
$$5x_1 - x_2 + 5x_3 = 9$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 9 \end{bmatrix}$$

Is division by zero an issue here? YES

$$12x_1 + 10x_2 - 7x_3 = 15$$
$$6x_1 + 5x_2 + 3x_3 = 14$$
$$24x_1 - x_2 + 5x_3 = 28$$

$$\begin{bmatrix} 12 & 10 & -7 \\ 6 & 5 & 3 \\ 24 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ 28 \end{bmatrix} \longrightarrow \begin{bmatrix} 12 & 10 & -7 \\ 0 & 0 & 6.5 \\ 12 & -21 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 6.5 \\ -2 \end{bmatrix}$$

Division by zero is a possibility at any step of forward elimination

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Exact Solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 6 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.9625 \\ 1.05 \\ 0.999995 \end{bmatrix}$$

Pitfall#2. Large Round-off Errors

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Solve it on a computer using 5 significant digits with chopping

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 1.5 \\ 0.99995 \end{bmatrix}$$

Is there a way to reduce the round off error?

Avoiding Pitfalls

Increase the number of significant digits

- Decreases round-off error
- Does not avoid division by zero

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

3. Gauss Elimination with Partial Pivoting

Avoiding Pitfalls

Gaussian Elimination with Partial Pivoting

- Avoids division by zero
- Reduces round off error

What is Different About Partial Pivoting?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots\ldots, |a_{nk}|$$

If the maximum of the values is $|a_{pk}|$

in the p^{th} row, $k \le p \le n$, then switch rows p and k.

Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Example (2^{nd} step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -7 & 6 & 1 & 2 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -17 & 12 & 11 & 43 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 8 \\ 9 \\ 3 \end{bmatrix}$$

Which two rows would you switch?

Example (2^{nd} step of FE)

$$\begin{bmatrix} 6 & 14 & 5.1 & 3.7 & 6 \\ 0 & -17 & 12 & 11 & 43 \\ 0 & 4 & 12 & 1 & 11 \\ 0 & 9 & 23 & 6 & 8 \\ 0 & -7 & 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 8 \\ 9 \\ -6 \end{bmatrix}$$

Switched Rows

Gaussian Elimination with Partial Pivoting

A method to solve simultaneous linear equations of the form [A][X]=[C]

Two steps

1. Forward Elimination
2. Back Substitution

Forward Elimination

Same as Gauss elimination method except that we switch rows before **each** of the (n-1) steps of forward elimination.

Example: Matrix Form at Beginning of 2nd Step of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ 0 & a'_{32} & a'_{33} & \cdots & a'_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a'_{n2} & a'_{n3} & a'_{n4} & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Matrix Form at End of Forward Elimination

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{'} & a_{23}^{'} & \cdots & a_{2n}^{'} \\ 0 & 0 & a_{33}^{''} & \cdots & a_{3n}^{''} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{'} \\ b_3^{''} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Back Substitution Starting Eqns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$

$$a''_{33}x_3 + \ldots + a''_{n}x_n = b''_3$$

$$\vdots \qquad \vdots$$

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

Back Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{\,i-1} - \sum_{j=i+1}^{n} a_{ij}^{\,i-1} x_j}{a_{ii}^{\,i-1}} \text{ for } i = n-1, \ldots, 1$$

Example 2

Solve the following set of equations by Gaussian elimination with partial pivoting

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example 2 Cont.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix}$$

1. Forward Elimination
2. Back Substitution

Forward Elimination

Number of Steps of Forward Elimination

Number of steps of forward elimination is

$$(n\text{-}1)=(3\text{-}1)=2$$

Forward Elimination: Step 1

- Examine absolute values of first column, first row and below.

$$|25|, |64|, |144|$$

- Largest absolute value is 144 and exists in row 3.
- Switch row 1 and row 3.

$$\begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 64 & 8 & 1 & \vdots & 177.2 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 64, $\frac{64}{144} = 0.4444$.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \times 0.4444 = \begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 & \vdots & 177.2 \end{bmatrix} \\ - \begin{bmatrix} 63.99 & 5.333 & 0.4444 & \vdots & 124.1 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix}$$

Divide Equation 1 by 144 and multiply it by 25, $\frac{25}{144} = 0.1736$.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \end{bmatrix} \times 0.1736 = \begin{bmatrix} 25.00 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 25 & 5 & 1 & \vdots & 106.8 \end{bmatrix} \\ - \begin{bmatrix} 25 & 2.083 & 0.1736 & \vdots & 48.47 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix}$$

Forward Elimination: Step 2

- Examine absolute values of second column, second row and below.

$$|2.667|, |2.917|$$

- Largest absolute value is 2.917 and exists in row 3.
- Switch row 2 and row 3.

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Forward Elimination: Step 2 (cont.)

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix}$$

Divide Equation 2 by 2.917 and multiply it by 2.667,

$$\frac{2.667}{2.917} = 0.9143.$$

$$\begin{bmatrix} 0 & 2.917 & 0.8264 & \vdots & 58.33 \end{bmatrix} \times 0.9143 = \begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & 2.667 & 0.5556 & \vdots & 53.10 \end{bmatrix} \\ -\begin{bmatrix} 0 & 2.667 & 0.7556 & \vdots & 53.33 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix}$$

Back Substitution

Back Substitution

$$\begin{bmatrix} 144 & 12 & 1 & \vdots & 279.2 \\ 0 & 2.917 & 0.8264 & \vdots & 58.33 \\ 0 & 0 & -0.2 & \vdots & -0.23 \end{bmatrix} \Rightarrow \begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_3

$$-0.2a_3 = -0.23$$

$$a_3 = \frac{-0.23}{-0.2}$$

$$= 1.15$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_2

$$2.917a_2 + 0.8264a_3 = 58.33$$

$$a_2 = \frac{58.33 - 0.8264a_3}{2.917}$$

$$= \frac{58.33 - 0.8264 \times 1.15}{2.917}$$

$$= 19.67$$

Back Substitution (cont.)

$$\begin{bmatrix} 144 & 12 & 1 \\ 0 & 2.917 & 0.8264 \\ 0 & 0 & -0.2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 279.2 \\ 58.33 \\ -0.23 \end{bmatrix}$$

Solving for a_1

$$144a_1 + 12a_2 + a_3 = 279.2$$

$$a_1 = \frac{279.2 - 12a_2 - a_3}{144}$$

$$= \frac{279.2 - 12 \times 19.67 - 1.15}{144}$$

$$= 0.2917$$

Gaussian Elimination with Partial Pivoting Solution

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 19.67 \\ 1.15 \end{bmatrix}$$

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_3}{2.5} = -1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Partial Pivoting: Example

Compare the calculated and exact solution

The fact that they are equal is coincidence, but it does illustrate the advantage of Partial Pivoting

$$X_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad X_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

4. Determinant of a Square Matrix Using Gauss Elimination

Example

Theorem of Determinants

If a multiple of one row of $[A]_{nxn}$ is added or subtracted to another row of $[A]_{nxn}$ to result in $[B]_{nxn}$ then $\det(A)=\det(B)$

Theorem of Determinants

The determinant of an upper triangular matrix $[A]_{nxn}$ is given by

$$\det A = a_{11} \times a_{22} \times \ldots \times a_{ii} \times \ldots \times a_{nn}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Forward Elimination of a Square Matrix

Using forward elimination to transform $[A]_{nxn}$ to an upper triangular matrix, $[U]_{nxn}$.

$$A_{n\times n} \rightarrow U_{n\times n}$$

$$\det(A) = \det(U)$$

Example

Using Gaussian elimination find the determinant of the following square matrix.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination

Forward Elimination: Step 1

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 64, $\frac{64}{25} = 2.56$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 2.56 = \begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix}$$

Subtract the result from Equation 2

$$\begin{array}{r} \begin{bmatrix} 64 & 8 & 1 \end{bmatrix} \\ -\begin{bmatrix} 64 & 12.8 & 2.56 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Forward Elimination: Step 1 (cont.)

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

Divide Equation 1 by 25 and multiply it by 144, $\frac{144}{25} = 5.76$.

$$\begin{bmatrix} 25 & 5 & 1 \end{bmatrix} \times 5.76 = \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 144 & 12 & 1 \end{bmatrix} \\ - \begin{bmatrix} 144 & 28.8 & 5.76 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Forward Elimination: Step 2

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Divide Equation 2 by −4.8

and multiply it by −16.8,

$$\frac{-16.8}{-4.8} = 3.5 \quad .$$

$$\begin{bmatrix} 0 & -4.8 & -1.56 \end{bmatrix} \times 3.5 = \begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix}$$

Subtract the result from Equation 3

$$\begin{array}{r} \begin{bmatrix} 0 & -16.8 & -4.76 \end{bmatrix} \\ -\begin{bmatrix} 0 & -16.8 & -5.46 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0.7 \end{bmatrix} \end{array}$$

Substitute new equation for Equation 3

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the Determinant

After forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{aligned} \det\ \mathrm{A} &= u_{11} \times u_{22} \times u_{33} \\ &= 25 \times \ -4.8 \ \times 0.7 \\ &= -84.00 \end{aligned}$$

5. Gaussian Elimination For Tridiagonial System

Tridiagonal Systems I

Introduction of Tridiagonal System?

- Special Linear System Arising in Application
- A general tridiagonal matrix is a matrix whose nonzero elements are found only on the diagonal, subdiagonal, and superdiagonal of the matrix.
- if $|i - j| > 1$,

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & . & . & . & & 0 \\ a_{21} & a_{22} & a_{23} & 0 & & & & . \\ 0 & a_{32} & a_{33} & a_{34} & 0 & & & . \\ . & 0 & a_{43} & a_{44} & . & . & & . \\ . & & 0 & . & . & . & . & \\ . & & & . & . & . & . & . \\ & & & & . & . & . & . \\ 0 & . & . & . & & . & . & a_{nn} \end{bmatrix}$$

Tridiagonal Systems II

- In Storage
  - Only the diagonal, subdiagonal, and superdiagonal elements of the general tridiagonal matrix are stored.

    This is called tridiagonal storage mode.
  - The elements of a general tridiagonal matrix, $\boldsymbol{A}$, of order n are stored in three one-dimensional arrays, C, D, and E, each of length n.
  - array C contains the subdiagonal elements, stored as follows:

    $C = (*, a21, a32, a43, \ldots, an,n\text{-}1)$
  - array D contains the main diagonal elements, stored as follows:

    $D = (a11, a22, a33, \ldots, ann)$
  - array E contains the superdiagonal elements, stored as follows:

    $E = (a12, a23, a34, \ldots, an\text{-}1,n, *)$
  - where "*" means you do not store an element in that position in the array

Tridiagonal Systems III

Example of Tridiagonal Matrix…

$$
\begin{aligned}
2x_1 \; -x_2 \qquad\qquad\qquad &= 1, \\
-x_1 + 2x_2 \; -x_3 \qquad\quad &= 0, \\
-x_2 + 2x_3 \; -x_4 &= 0, \\
-x_3 + 2x_4 &= 1.
\end{aligned}
$$

$$
\left[\begin{array}{cccc|c}
2 & -1 & 0 & 0 & 1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & 1
\end{array}\right]
$$

Gaussian Elimination for Tridiagonal Systems

$$\begin{aligned} 2x_1 - x_2 &= 1, \\ -x_1 + 2x_2 - x_3 &= 0, \\ -x_2 + 2x_3 - x_4 &= 0, \\ -x_3 + 2x_4 &= 1. \end{aligned}$$

$$\begin{aligned} x_1 - \frac{1}{2}x_2 &= \frac{1}{2}, \\ \frac{3}{2}x_2 - x_3 &= \frac{1}{2}, \\ -x_2 + 2x_3 - x_4 &= 0, \\ -x_3 + 2x_4 &= 1. \end{aligned}$$

$$\begin{aligned} x_1 - \frac{1}{2}x_2 &= \frac{1}{2}, \\ x_2 - \frac{2}{3}x_3 &= \frac{1}{3}, \\ +x_3 - \frac{3}{4}x_4 &= \frac{1}{4}, \\ x_4 &= 1. \end{aligned}$$

$$\begin{aligned} x_4 &= 1; \\ x_3 &= \frac{1}{4} - \left(-\frac{3}{4}\right)(1) = 1; \\ x_2 &= \frac{1}{3} - \left(-\frac{2}{3}\right)(1) = 1; \\ x_1 &= \frac{1}{2} - \left(-\frac{1}{2}\right)(1) = 1. \end{aligned}$$

Solving a Tridiagonal Systems Using the Thomas Method I

$$
\begin{aligned}
d_1x_1 + a_1x_2 \quad & = r_1, \\
b_2x_1 + d_2x_2 + a_2x_3 \quad \ldots. \quad & = r_2, \\
+ b_{n-1}x_{n-2} + d_{n-1}x_{n-1} + a_{n-1}x_n \quad & = r_{n-1}, \\
+ b_nx_{n-1} + d_nx_n \quad & = r_n.
\end{aligned}
$$

1. B_1 and A_n are zero.
2. This algorithm takes advantage of the zero elements that are already present in the coefficient matrix and avoids unnecessary arithmetic operations.

   Thus, we need to store only the new vectors a and r.

Solving a Tridiagonal Systems Using the Thomas Method II

❖ Step 1 : For the first equation

$$a_1 = \frac{a_1}{d_1}, r_1 = \frac{r_1}{d_1}$$

❖ Step 2 : For each of the equation

$$a_i = \frac{a_i}{d_i - b_i a_{i-1}}, r_i = \frac{r_i - b_i r_{i-1}}{d_i - b_i a_{i-1}}$$

❖ Step 3 : For the last equation

$$r_n = \frac{r_n - b_n r_{n-1}}{d_n - b_n a_{n-1}}$$

❖ Step 4 : by back substitution

$$x_n = r_n,$$

$$x_i = r_i - a_i x_{i+1},$$

$$i = n-1, n-2, n-3, \ldots, 2, 1.$$

Solving a Tridiagonal Systems Using the Thomas Method III

$$
\begin{aligned}
2x_1 - x_2 &= 1,\\
-x_1 + 2x_2 - x_3 &= 0,\\
-x_2 + 2x_3 - x_4 &= 0,\\
-x_3 + 2x_4 &= 1.
\end{aligned}
$$

$$d = (2,2,2,2); \qquad a = (-1,-1,-1,0);$$

$$b = (0,-1,-1,-1); \qquad r = (1,0,0,1).$$

Solving a Tridiagonal Systems Using the Thomas Method IV

$$a_1 = \frac{a_1}{d_1} = -\frac{1}{2}, r_1 = \frac{r_1}{d_1} = \frac{1}{2}.$$

$$a_2 = \frac{a_2}{d_2 - b_2 a_1} = \frac{-1}{2-(-1)(-1/2)} = -\frac{2}{3},$$

$$r_2 = \frac{r_2 - b_2 r_1}{d_2 - b_2 a_1} = \frac{0-(-1)(1/2)}{2-(-1)(-1/2)} = \frac{1}{3}.$$

$$x_4 = r_4 = 1,$$

$$x_3 = r_3 - a_3 x_4 = 1/4 - (-3/4) = 1,$$

$$x_2 = r_2 - a_2 x_3 = 1/3 - (-2/3)(1) = 1,$$

$$x_1 = r_1 - a_1 x_2 = 1/2 - (-1/2)(1) = 1.$$

Discussion of Thomas method

*** The required multiplications and divisions for Thomas method.**

For the first equation, 2divisions are needed.

For each of the next n-2 equations, 2multiplications and 2 divisions are needed.

For the last equation, 2 multiplications and 1 division are required.

The total for elimination is 5+4(n-2).

For the back substitution, n-1 multiplications are needed.

* The Thomas algorithm requires that $d_1 \neq 0$ and $d_i - b_i a_{i-1} \neq 0$ for each i.

Using the Thomas Method for a System that Would Require Pivoting for Gaussian Elimination

I

$$
\begin{aligned}
2x_1 - x_2 &= 1, \\
-x_1 + 2x_2 - x_3 &= 0, \\
-x_2 + \frac{2}{3}x_3 - x_4 &= -\frac{4}{3}, \\
-x_3 + 2x_4 - x_5 &= 0, \\
-x_4 + 2x_5 - x_6 &= 0, \\
-x_5 + 2x_6 &= 1.
\end{aligned}
$$

$$
\begin{aligned}
x_1 - \frac{1}{2}x_2 &= \frac{1}{2}, \\
+x_2 - \frac{2}{3}x_3 &= \frac{1}{3}, \\
-x_4 &= -1, \\
-x_3 + 2x_4 - x_5 &= 0, \\
-x_4 + 2x_5 - x_6 &= 0, \\
-x_5 + 2x_6 &= 1.
\end{aligned}
$$

Using the Thomas Method for a System that Would Require Pivoting for Gaussian Elimination II

$$
\begin{aligned}
x_1 - \frac{1}{2}x_2 &= \frac{1}{2}, \\
+x_2 - \frac{2}{3}x_3 &= \frac{1}{3}, \\
-x_4 &= -1(solve), \\
-x_3 \quad +2x_4 \quad -x_5 &= 0(skip \quad for \quad now), \\
+2x_5 \quad -x_6 &= 1(u\sin g x_4 = 1), \\
-x_5 + 2x_6 &= 1.
\end{aligned}
$$

Using the Thomas Method for a System that Would Require Pivoting for Gaussian Elimination III

$$
\begin{aligned}
x_1 - \frac{1}{2}x_2 \quad &= \frac{1}{2},\\
+x_2 - \frac{2}{3}x_3 \quad &= \frac{1}{3},\\
-x_4 \quad &= -1,\\
-x_3 + 2x_4 - x_5 \quad &= 0,\\
x_5 - \frac{1}{2}x_6 &= \frac{1}{2},\\
x_6 &= 1.
\end{aligned}
$$

$$
\begin{aligned}
x6 &= 1,\\
x_5 &= \frac{1}{2} + \frac{1}{2}x_6 = 1,\\
x_4 &= 1,\\
x_3 &= 2x_4 - x_5 = 1,\\
x_2 &= \frac{1}{3} + \frac{2}{3}x_3 = 1,\\
x_1 &= \frac{1}{2} + \frac{1}{2}x_2 = 1.
\end{aligned}
$$

6. Gauss – Jordan Elimination

Gauss-Jordan

- Variation of Gauss elimination
- Primary motive for introducing this method is that it provides a simple and convenient method for computing the *matrix inverse.*
- When an unknown is eliminated, it is eliminated from all other equations, rather than just the subsequent one.
- All rows are normalized by dividing them by their pivot elements
- Elimination step results in an identity matrix

Introduction

- For inverting a matrix, ***Gauss-Jordan elimination*** is about as efficient as any other method. For solving sets of linear equations, Gauss-Jordan elimination produces *both* the solution of the equations for one or more right-hand side vectors $\boldsymbol{b}$, and also the matrix inverse $\boldsymbol{A}^{-1}$.
- However, its principal weaknesses are
  - (i) that it requires all the right-hand sides to be stored and manipulated at the same time, and
  - (ii) that when the inverse matrix is *not* desired, Gauss-Jordan is three times slower than the best alternative technique for solving a single linear set
- The method's principal strength is that it is as stable as any other direct method, perhaps even a bit more stable when full pivoting is used

Graphical depiction of Gauss-Jordan

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & c_1 \\ a_{21} & a_{22}^{©} & a_{23}^{©} & | & c_2^{©} \\ a_{31} & a_{32} & a_{33}^{©©} & | & c_3^{©©} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & | & c_1^{n} \\ 0 & 1 & 0 & | & c_2^{n} \\ 0 & 0 & 1 & | & c_3^{n} \end{bmatrix}$$

Graphical depiction of Gauss-Jordan

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & c_1 \\ a_{21} & a'_{22} & a'_{23} & | & c'_2 \\ a_{31} & a_{32} & a''_{33} & | & c''_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & | & c_1^{n} \\ 0 & 1 & 0 & | & c_2^{n} \\ 0 & 0 & 1 & | & c_3^{n} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & c_1^{n} \\ 0 & 1 & 0 & | & c_2^{n} \\ 0 & 0 & 1 & | & c_3^{n} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{aligned} x_1 &= c_1^{n} \\ x_2 &= c_2^{n} \\ x_3 &= c_3^{n} \end{aligned}$$

Gauss-Jordan Elimination

Let us consider the set of linearly independent equations.

$$\begin{cases} 2x - 4y + 5z = 36 \\ -3x + 5y + 7z = 7 \\ 5x + 3y - 8z = -31 \end{cases}$$

Augmented matrix for the set is:

$$\left[\begin{array}{ccc|c} 2 & -4 & 5 & 36 \\ -3 & 5 & 7 & 7 \\ 5 & 3 & -8 & -31 \end{array}\right]$$

Gauss-Jordan Elimination

Step 1: Eliminate x from the 2nd and 3rd equation.

$$\begin{bmatrix} 2 & -4 & 5 & 36 \\ -3 & 5 & 7 & 7 \\ 5 & 3 & -8 & -31 \end{bmatrix} \qquad \begin{cases} 2x-4y+5z=36 \\ -3x+5y+7z=7 \\ 5x+3y-8z=-31 \end{cases}$$

$$R_2' \Leftarrow \frac{3}{2}R_1 + R_2$$

$$R_3' \Leftarrow -\frac{5}{2}R_1 + R_3$$

$$\begin{bmatrix} 2 & -4 & 5 & 36 \\ 0 & -1 & 14.5 & 61 \\ 0 & 13 & -20.5 & -121 \end{bmatrix} \qquad \begin{cases} 2x-4y+5z=36 \\ -y+14.5z=61 \\ 13y-20.5z=-121 \end{cases}$$

Gauss-Jordan Elimination

Step 2: Eliminate y from the 3rd equation.

$13R'_2 + R'_3 \rightarrow R''_3$

$$\begin{bmatrix} 2 & -4 & 5 & 36 \\ 0 & -1 & 14.5 & 61 \\ 0 & 0 & 168 & 672 \end{bmatrix} \Rightarrow \begin{cases} 2x - 4y + 5z = 36 \\ -y + 14.5z = 61 \\ 168z = 672 \end{cases}$$

Step 3:

$0.5R'_1 \rightarrow R'_1$

$-R'_2 \rightarrow R''_2$

$(1/168)R''_3 \rightarrow R'''_3$

$$\begin{bmatrix} 1 & -2 & 2.5 & 18 \\ 0 & 1 & -14.5 & -61 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{cases} x - 2y + 2.5z = 18 \\ y - 14.5z = -61 \\ z = 4 \end{cases}$$

Gauss-Jordan Elimination

Step 4: Eliminate z from the 2nd equation

$$\begin{bmatrix} 1 & -2 & 2.5 & 18 \\ 0 & 1 & -14.5 & -61 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{cases} x-2y+2.5z=18 \\ y-14.5z=-61 \\ z=4 \end{cases}$$

$$(Row\,3)\times(14.5)+(Row\,2)$$

$$\begin{bmatrix} 1 & -2 & 2.5 & 18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{cases} x-2y+2.5z=18 \\ y=-3 \\ z=4 \end{cases}$$

Gauss-Jordan Elimination

Step 5-1: Eliminate y from the 1st equation

$$\begin{bmatrix} 1 & -2 & 2.5 & 18 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{cases} x - 2y + 2.5z = 18 \\ y = -3 \\ z = 4 \end{cases}$$

$(Row\,2) \times (2) + (Row\,1) \Rightarrow New\ Row1$

$$\begin{bmatrix} 1 & 0 & 2.5 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{cases} x + 2.5z = 12 \\ y = -3 \\ z = 4 \end{cases}$$

Gauss-Jordan Elimination

Step 5-2: Eliminate z from the 1st equation

$$\begin{bmatrix} 1 & 0 & 2.5 & 12 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{cases} x + 2.5z = 12 \\ y = -3 \\ z = 4 \end{cases}$$

$$(Row\,3) \times (-2.5) + (Row\,1) \Rightarrow New\ Row1$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{cases} x = 2 \\ y = -3 \\ z = 4 \end{cases}$$

Example

Given the following, determine {x} for the two different loads {c}

$$Ax = c$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 6 & 3 \\ -3 & 1 & -4 \end{bmatrix}$$

$$c^{T} = \begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$c^{T} = \begin{matrix} 4 & -7 & 1 \end{matrix}$$

Solution

$$Ax = c$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 6 & 3 \\ -3 & 1 & -4 \end{bmatrix}$$

$$c^{T} = 1 \quad 2 \quad 3$$

$$c^{T} = 4 \quad -7 \quad 1$$

$\{c\}^T = \{1 \quad 2 \quad 3\}$

$x_1 = (2)(1) + (-1)(2) + (1)(3) = 3$

$x_2 = (-2)(1) + (6)(2) + (3)(3) = 19$

$x_3 = (-3)(1) + (1)(2) + (-4)(3) = -13$

$\{c\}^T = \{4 \quad -7 \quad 1)$

$x_1 = (2)(4) + (-1)(-7) + (1)(1) = 16$

$x_2 = (-2)(4) + (6)(-7) + (3)(1) = -47$

$x_3 = (-3)(4) + (1)(-7) + (-4)(1) = -23$