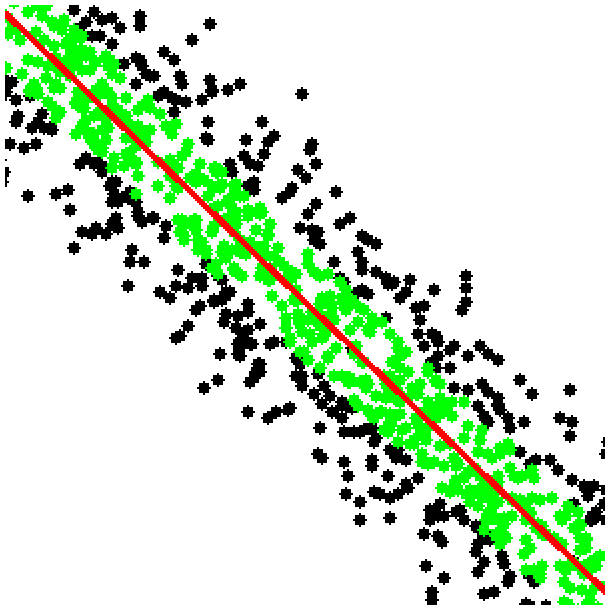
# Robust Mechatronics

## Handling Measurement Noise

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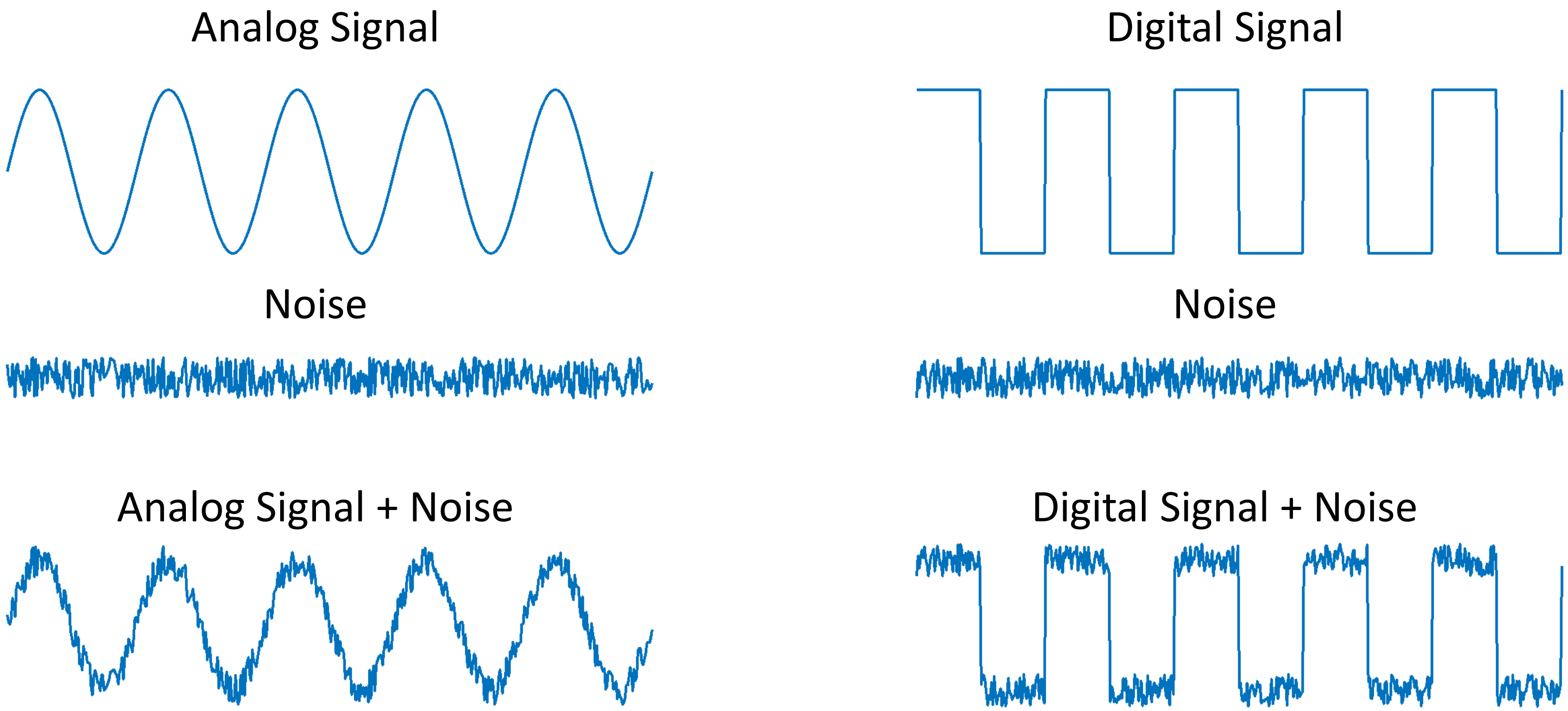
Mechatronics & Systems Automation Lab

# Handling Measurement Noise

***Noise***

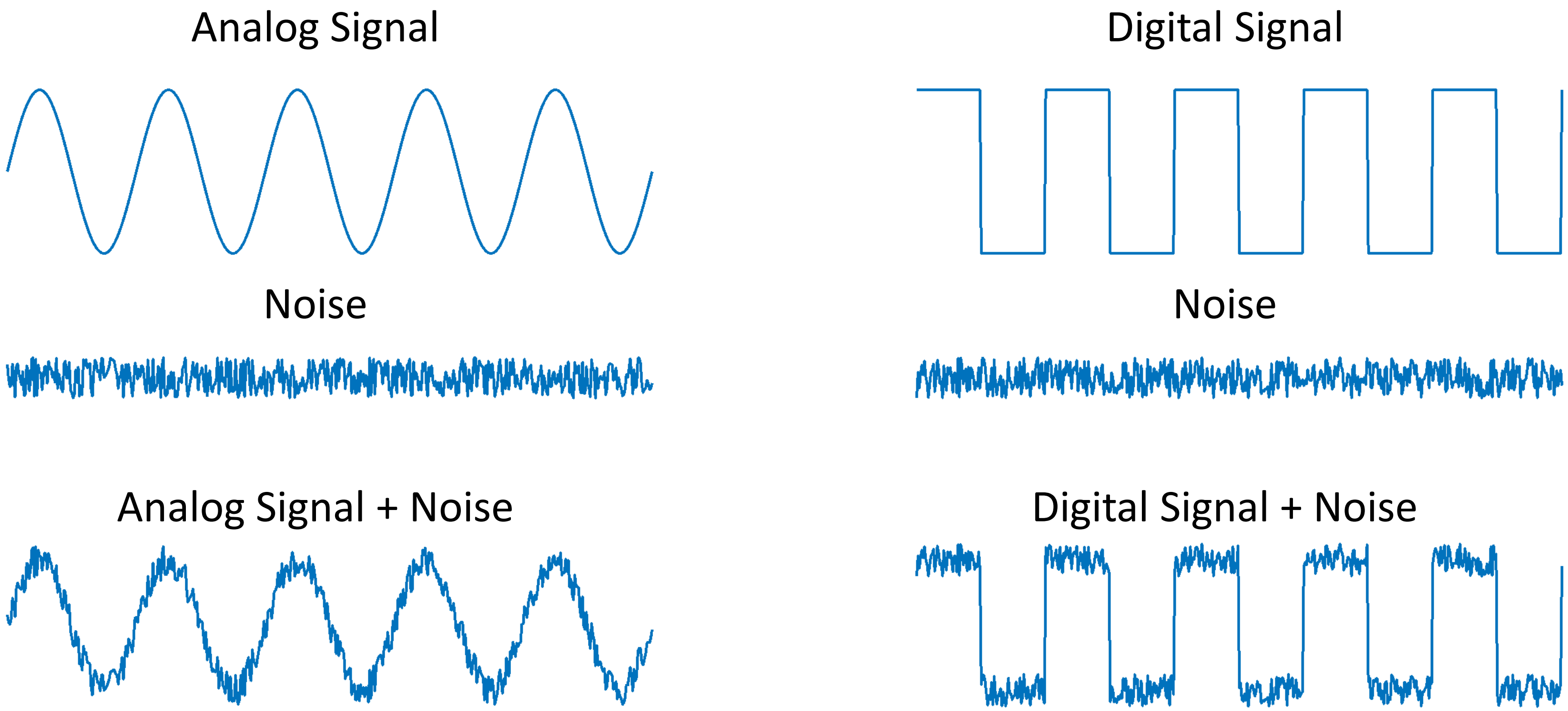
- All measurements are influenced by extraneous factors that are unrelated to the specific quantity being measured



# Handling Measurement Noise

***Noise***

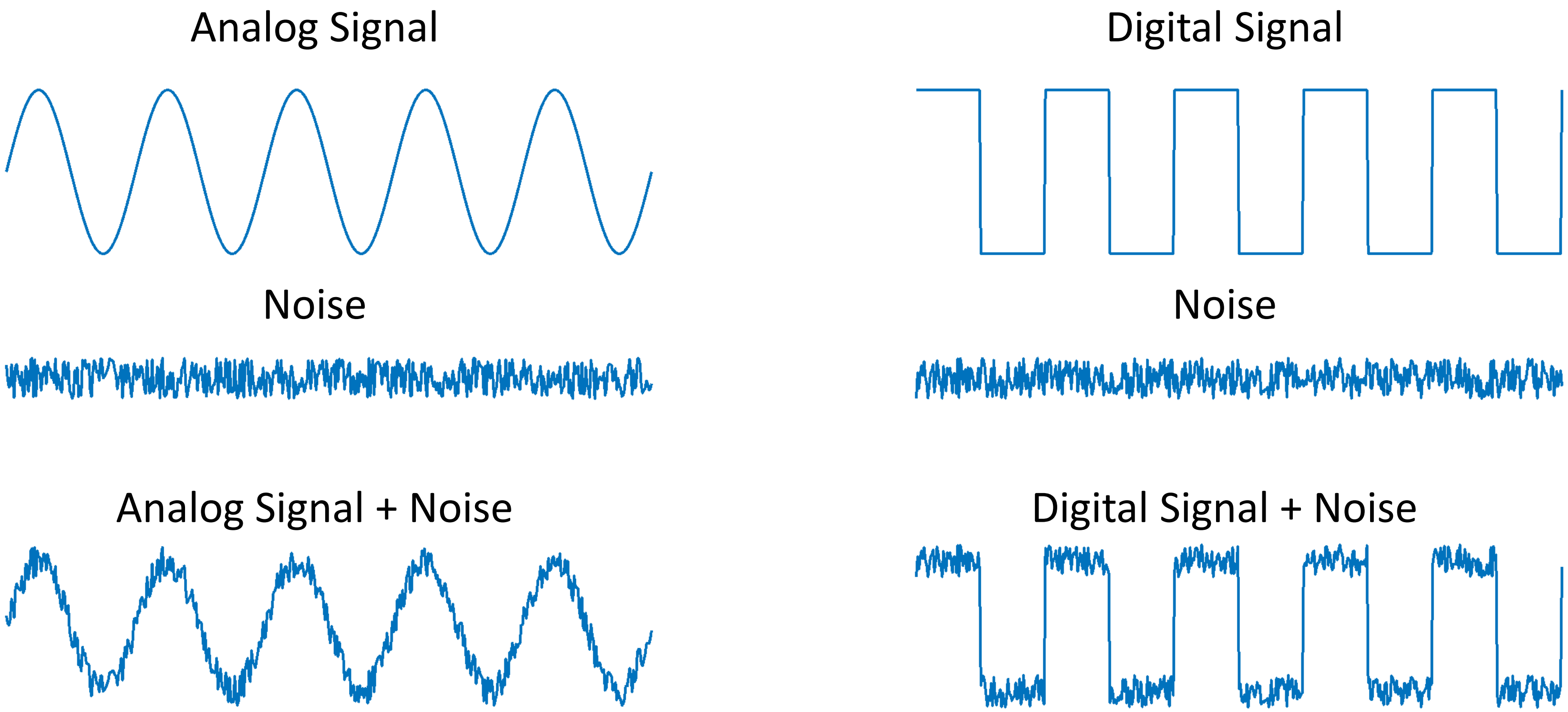
- Signal noise refers to time-dependent fluctuations in the instrument's output that occur randomly (or near-randomly) and are not attributable to the analyte's presence or response



# Handling Measurement Noise

***Noise***

- Assessing the magnitude of noise relative to the magnitude of the signal is essential for evaluating measurement accuracy and establishing the minimum detectable signal level, known as the detection limit



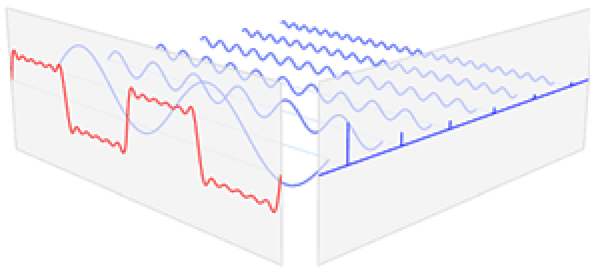
# Handling Measurement Noise

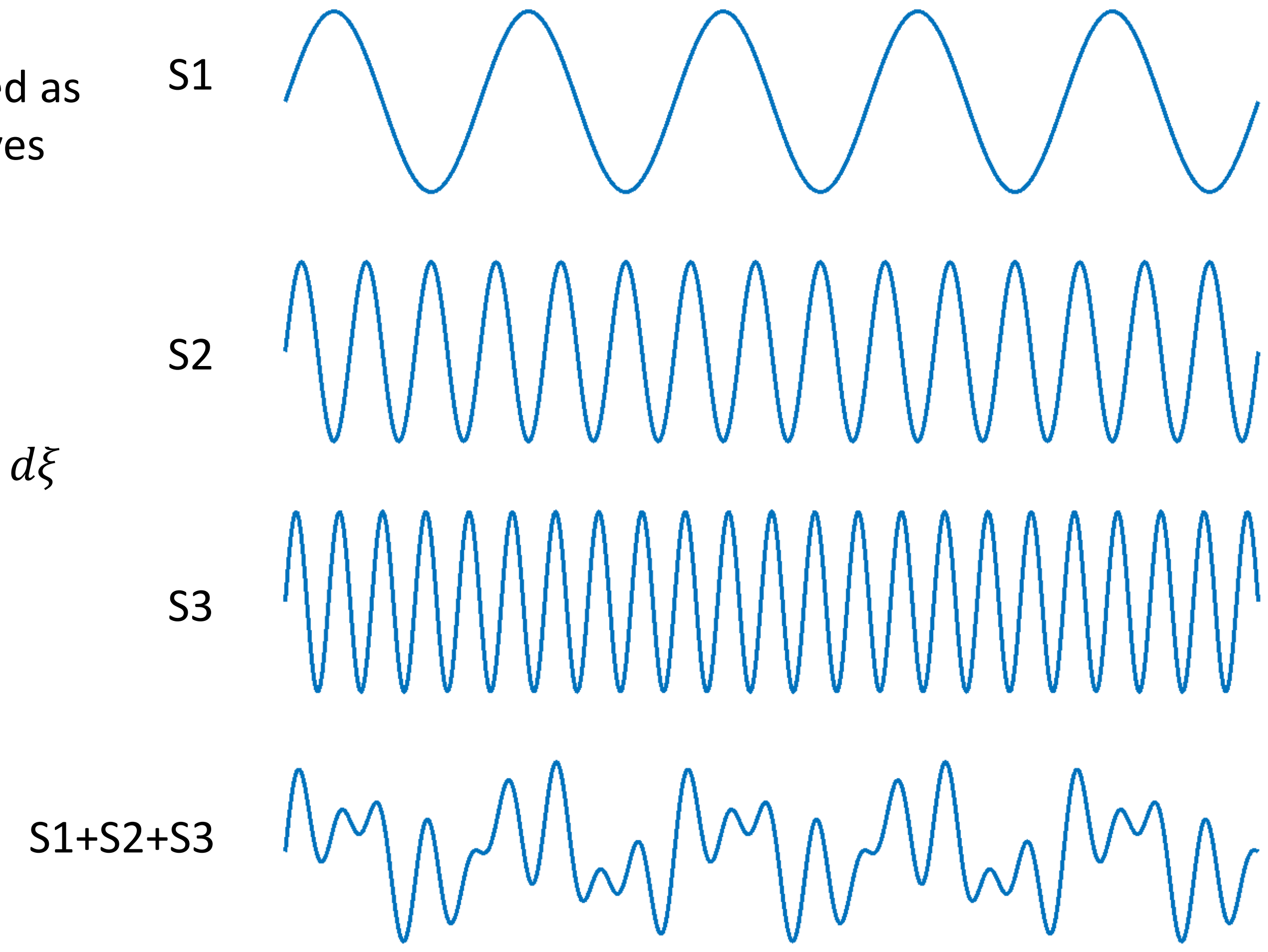
***Noise***

- Noise arises from many sources, and it can be treated as a sine wave, or the sum of multiple/infinite sine waves

*~Fourier theory*

$$f(t) = \int_{-\infty}^{+\infty} \hat{f}^{s}(\xi)\sin(2\pi\xi t)\,d\xi + \int_{-\infty}^{+\infty} \hat{f}^{c}(\xi)\cos(2\pi\xi t)\,d\xi$$

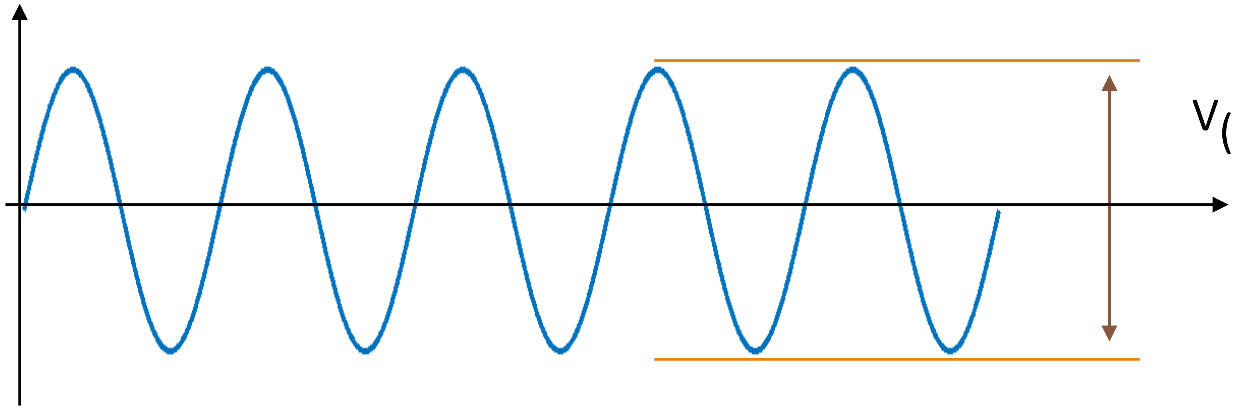




# Handling Measurement Noise

***Noise***

- Peak-to-peak amplitude



$V_{(peak-to-peak)}$

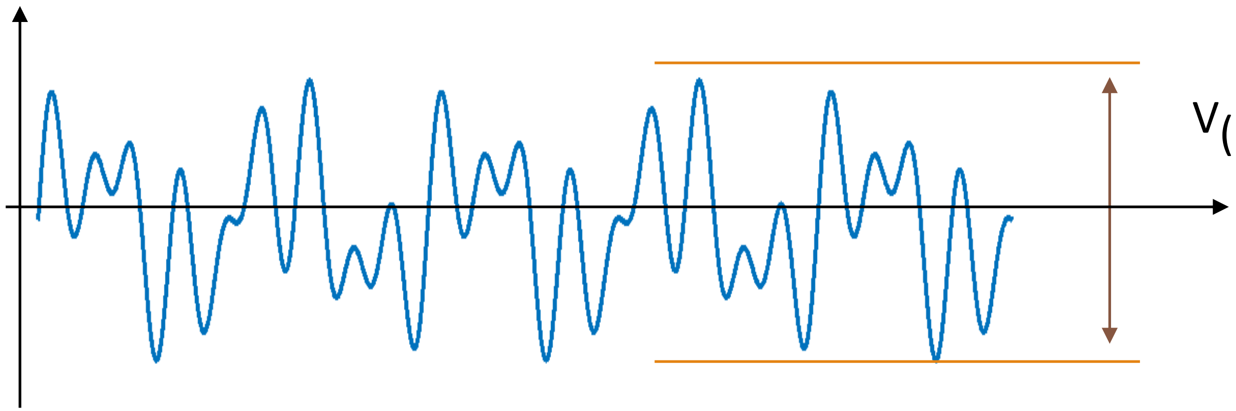
*or*

$V_{(p-p)}$

# Handling Measurement Noise

***Noise***

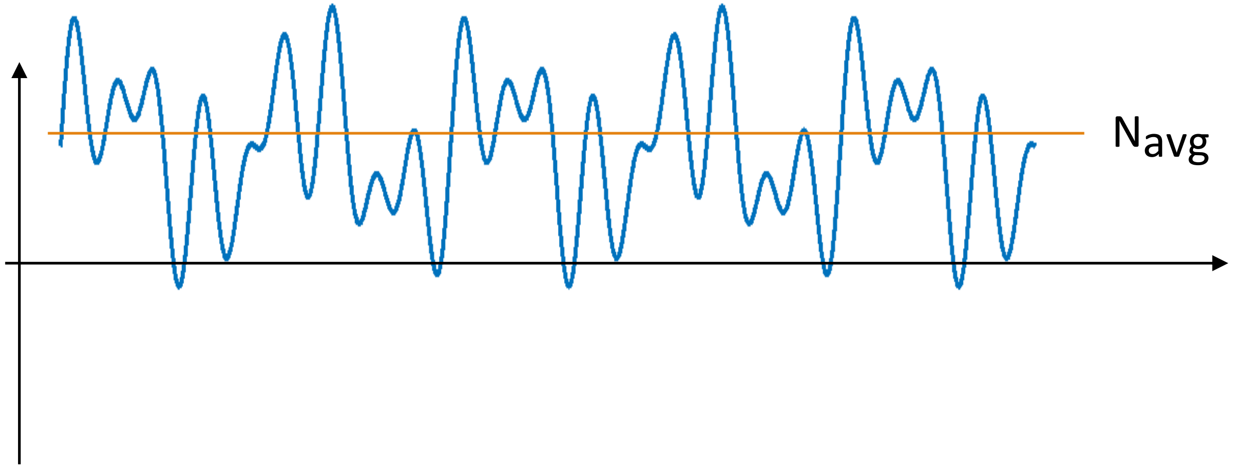
- Peak-to-peak amplitude for noise



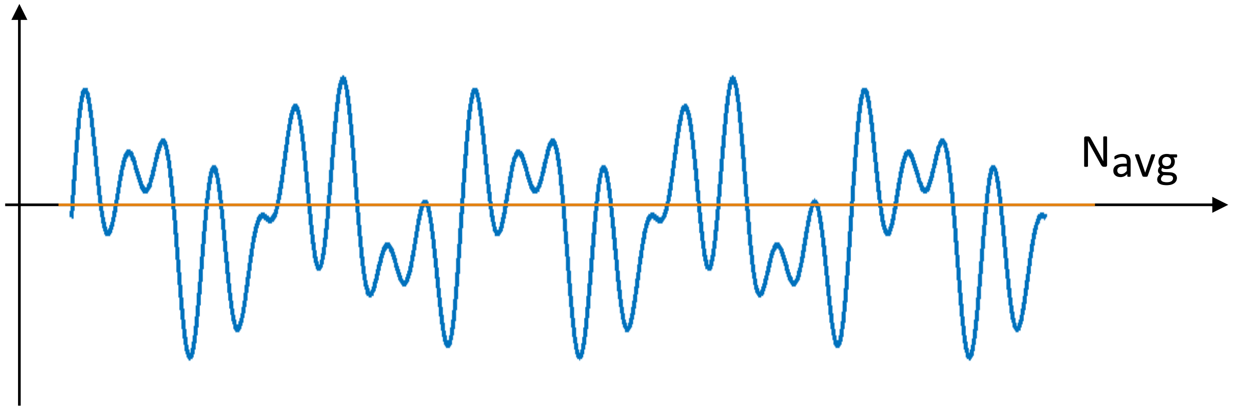
# Handling Measurement Noise

***Noise***

- Average noise




- However, if noise is truly random,
  it is possible that $N_{avg} = 0$.
  Therefore, Average Noise is not explicitly useful.




- If $N_{avg} \neq 0$, then another (constant) noise parameter is present, which can be identified, measured, and eliminated (DC constant).

# Handling Measurement Noise

***Noise***

- Why $N_{avg}$ can become zero?
  - Equally possible and similar amplitude positive and negative values

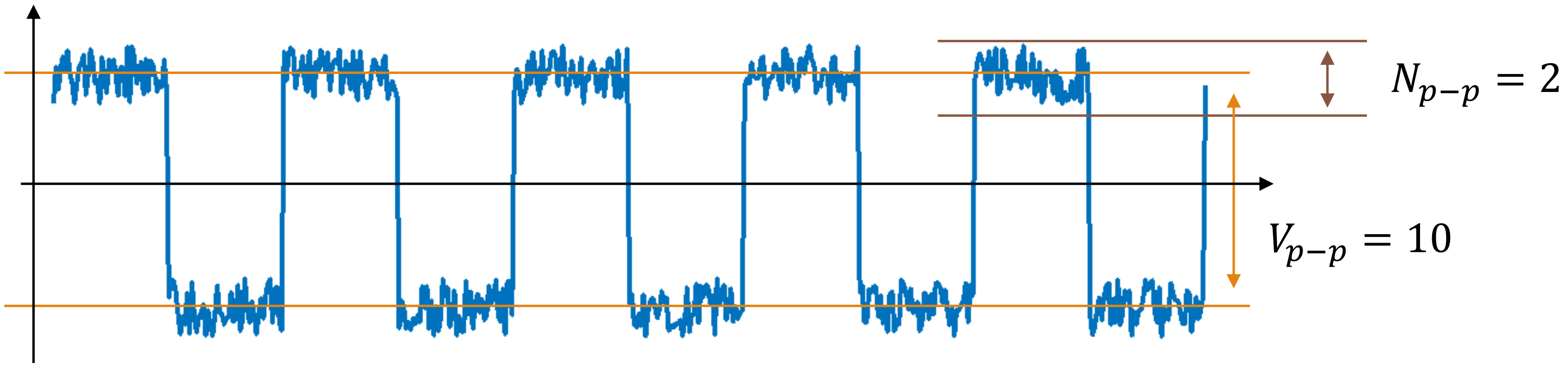
- Solution?
  - Squaring the signal → Root Mean Square (RMS)

$$N_{RMS} = \lim_{T\to\infty} \sqrt{\frac{1}{2T}\int_{-T}^{T} [N(t)]^2 \, dt} \quad \xrightarrow{\text{for sine wave}} \quad N_{RMS} = \frac{1}{2\sqrt{2}} N_{p-p} \quad \rightarrow \quad N_{RMS} \approx 0.35 \, N_{p-p}$$

# Handling Measurement Noise

***Noise***

- Signal-to-noise ratio



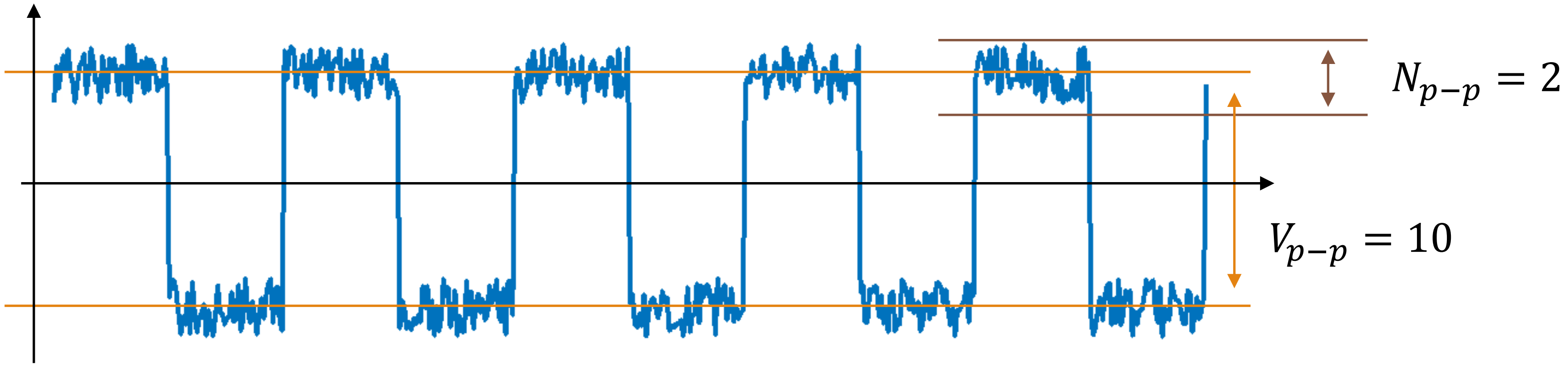

$N_{RMS} = 0.35\ N_{p-p} = 0.7$

$V_{RMS} = V_{p-p}/2 = 5$

$S/N = 5/0.7 \approx 7.14$

# Handling Measurement Noise

***Noise***



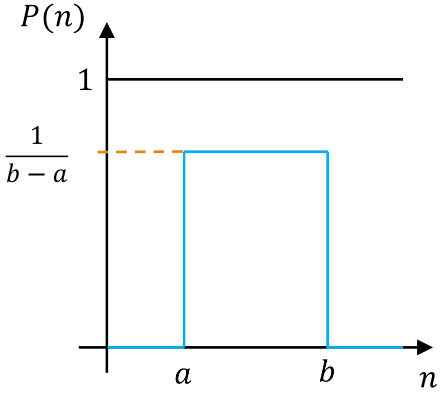

- We wish for the $S/N$ to be high enough in order to identify correct stimulus
  - A rule of thumb is for the $S/N$ to be at least 3

# Handling Measurement Noise

## ***Noise***

- The probability distribution of noise can be measured and modeled

### Uniform Noise

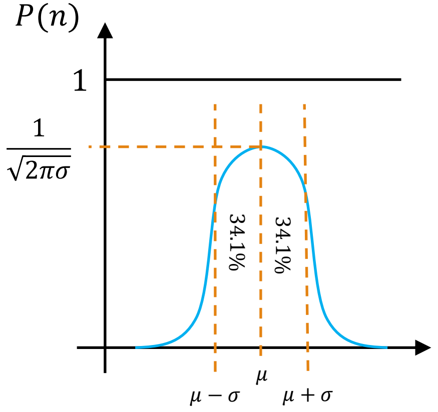



$$P(n) = \begin{cases} \frac{1}{b-a}, & for\ a \leq n \leq b \\ 0, & for\ n < a\ or\ n > b \end{cases}$$

$$E[n] = \mu = \frac{b+a}{2}$$

$$E\left[(n-\mu)^2\right] = \sigma^2 = \frac{(b-a)^2}{12}$$

### Gaussian Noise

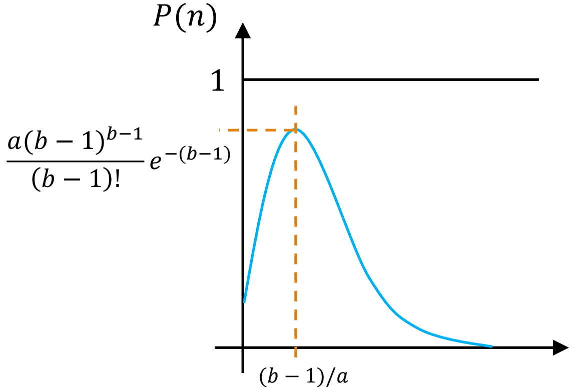



$$P(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(n-\mu)^2}{2\sigma^2}}$$

$$E[n] = \mu$$

$$E\left[(n-\mu)^2\right] = \sigma^2$$

### Gamma Noise

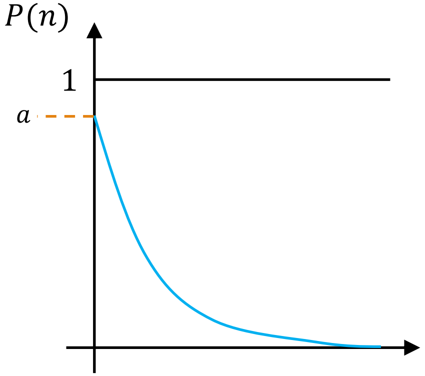



$$P(n) = \begin{cases} \frac{a^b n^{b-1}}{(b-1)!} e^{-an}, & for\ n \geq 0 \\ 0, & for\ n < 0 \end{cases}$$

$$E[n] = \mu = \frac{b}{a}$$

$$E\left[(n-\mu)^2\right] = \sigma^2 = \frac{b}{a^2}$$

### Exponential Noise




$$P(n) = \begin{cases} ae^{-an}, & for\ n \geq 0 \\ 0, & for\ n < 0 \end{cases}$$

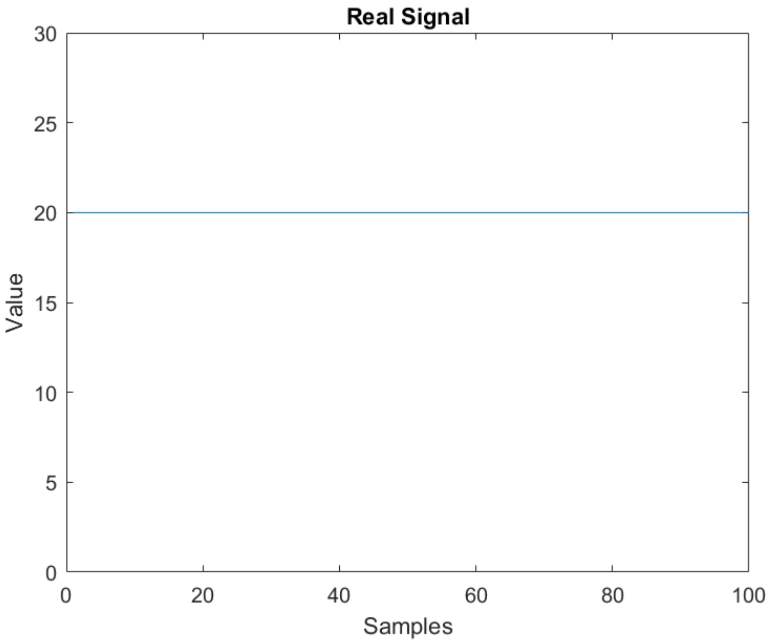
$$E[n] = \mu = \frac{1}{a}$$

$$E\left[(n-\mu)^2\right] = \sigma^2 = \frac{1}{a^2}$$

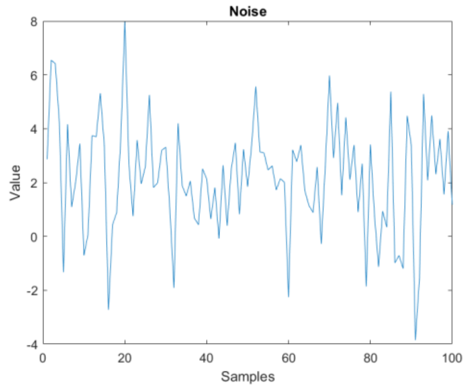
# Handling Measurement Noise

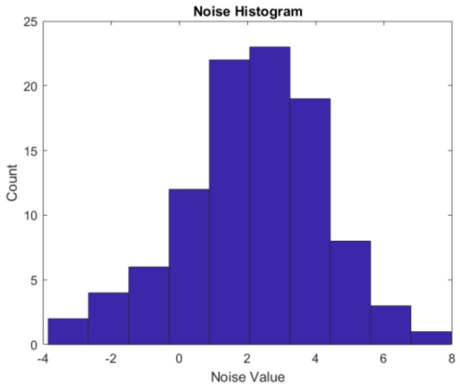
***Handling Noise***

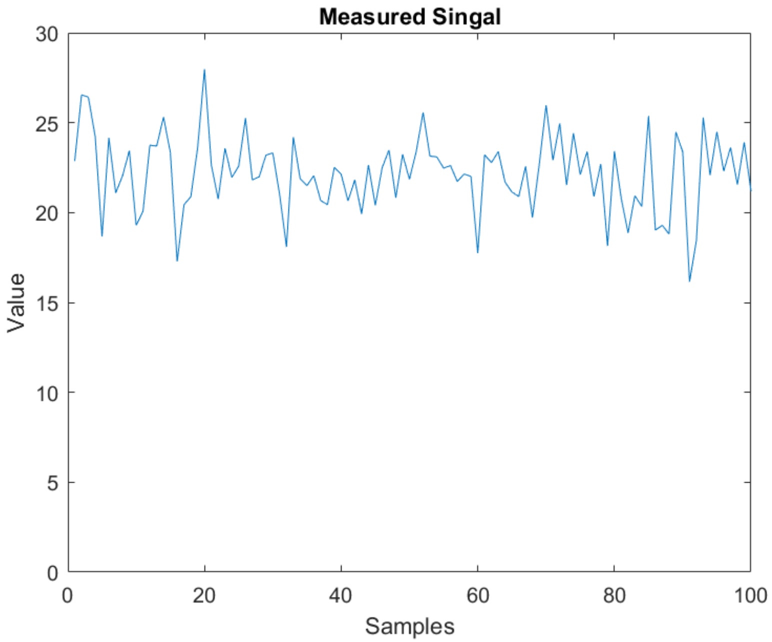
- Knowing the noise distribution yields some interesting properties for improving the accuracy of our measurements




$\mu_n = 2, \sigma_n = 2$




$\mu_m = 22.1689$

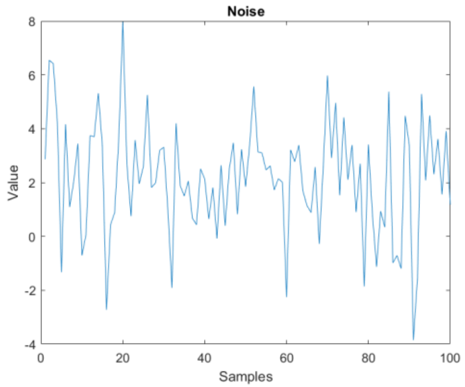
$S = \mu_m - \mu_n = 20.1689$

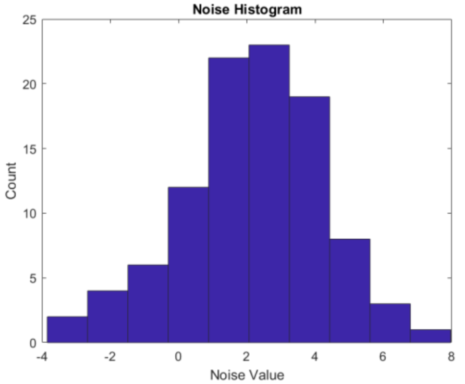
# Handling Measurement Noise

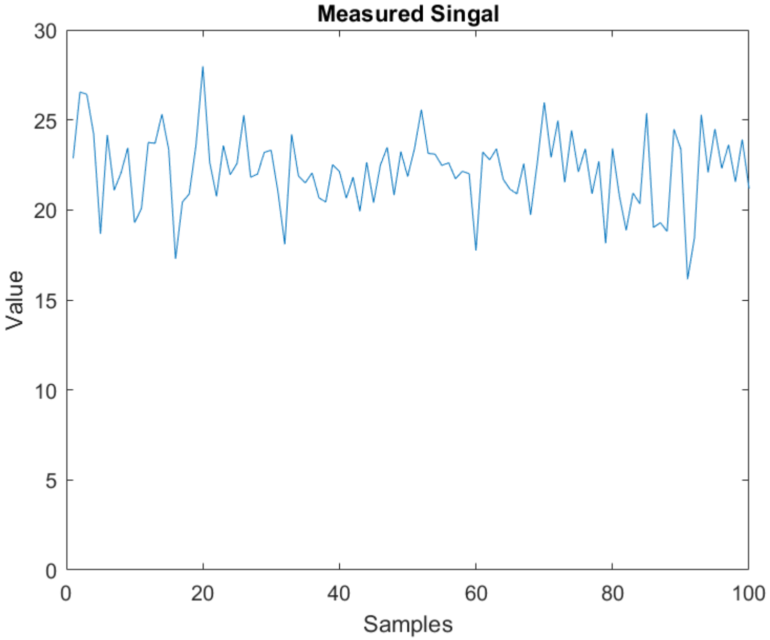
## ***Handling Noise***

- How many measurements should I take?     Answer: how certain do I need to be?

$\mu_n = 2, \sigma_n = 2$

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# Handling Measurement Noise

***Handling Noise***

- How many measurements should I take?

**Annex:**

The variance of a random variable $X$ is the expected value of the squared deviation from the mean of $X$, $\mu = E[X]$:

$$Var(X) = E[X - \mu^2]$$

**Discrete random variable**

$$Var(X) = \sum_{i=1}^{n} p_i (x_i - \mu)^2$$

$$\mu = E[X] = \sum_{i=1}^{n} p_i x_i$$

**Absolutely continuous random variable**

$$Var(X) = \int_{\mathbb{R}} x^2 f(x) dx - \mu^2$$

$$\mu = E[X] = \int_{\mathbb{R}} x f(x) dx$$

**Properties**

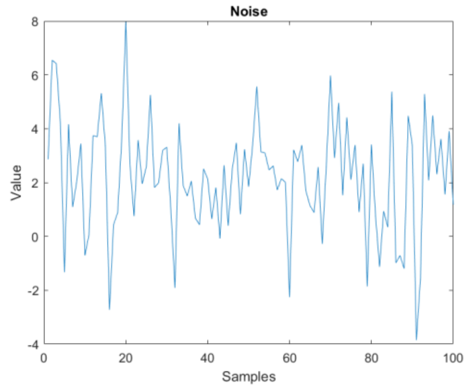
- $Var(X + a) = Var(X)$
- $Var(aX) = a^2 Var(X)$

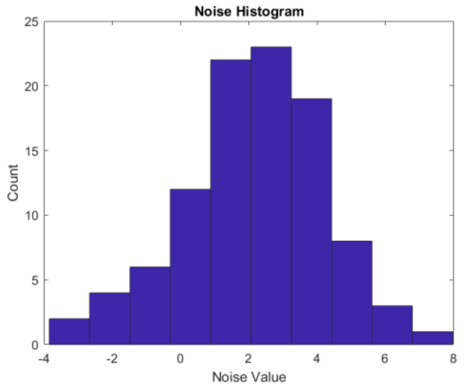
# Handling Measurement Noise

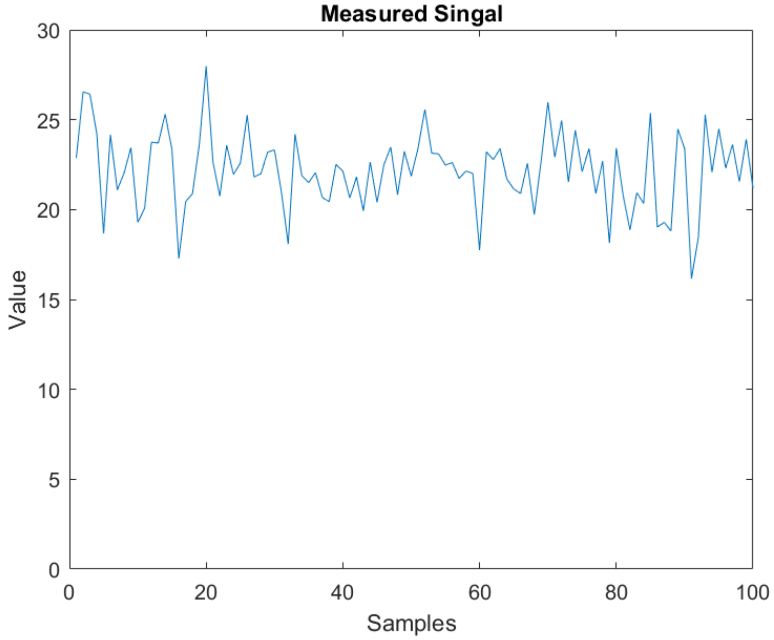
## ***Handling Noise***

- How many measurements should I take? Answer: how certain do I need to be?

$\mu_n = 2, \sigma_n = 2$




$\mu_m = 22.1689$

$S = \mu_m - \mu_n = 20.1689$

Let $X_1, X_2, \ldots, X_N$ be **independent** and **identically distributed** random variables, each with:

- $E[X_i] = \mu$
- $\text{Var}(X_i) = \sigma^2_{noise}$

The sample mean $\bar{X}$ is:

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

The mean value calculated from the sample will have an associated standard error on the mean:

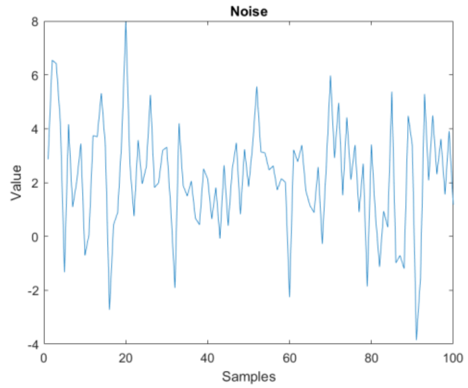
$$\sigma^2_{mean} = \text{Var}(\bar{X}) = Var\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) = \frac{1}{N^2} Var\left(\sum_{i=1}^{N} X_i\right)$$

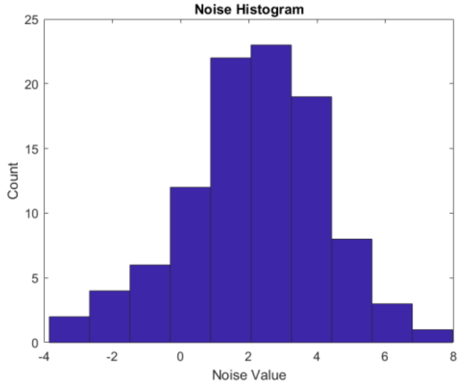
# Handling Measurement Noise

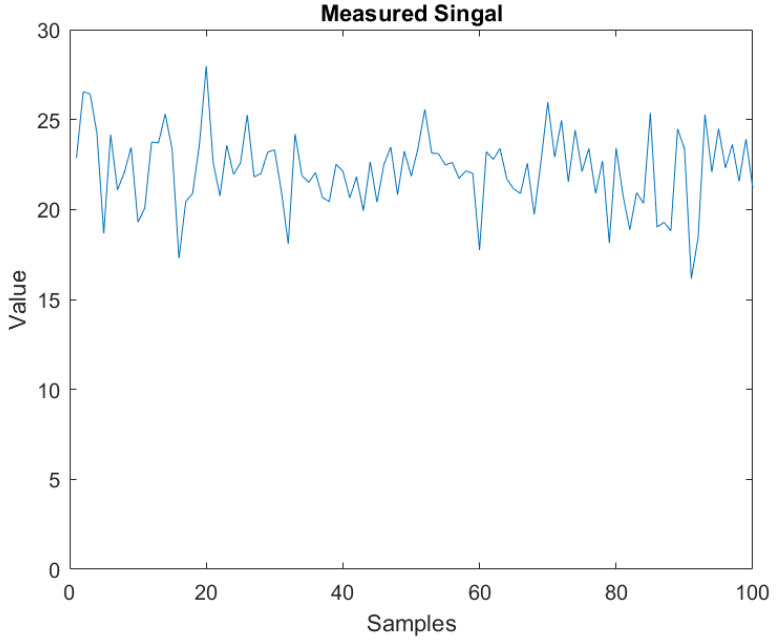
## ***Handling Noise***

- How many measurements should I take? Answer: how certain do I need to be?

$\mu_n = 2, \sigma_n = 2$




$\mu_m = 22.1689$

$S = \mu_m - \mu_n = 20.1689$

$$\mathrm{Var}(\bar{X}) = \frac{1}{N^2} Var\left(\sum_{i=1}^{N} X_i\right)$$

$$E[X_i] = \mu$$

$$\mathrm{Var}(X_i) = \sigma_{noise}^2$$

Since the $X_i$ are **independent**, the variance of the sum is the sum of the variances:

$$Var\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} Var(X_i) = N\sigma_{noise}^2$$

Thus,

$$\mathrm{Var}(\bar{X}) = \frac{1}{N^2} N\sigma_{noise}^2 = \frac{\sigma_{noise}^2}{N}$$

Finally,

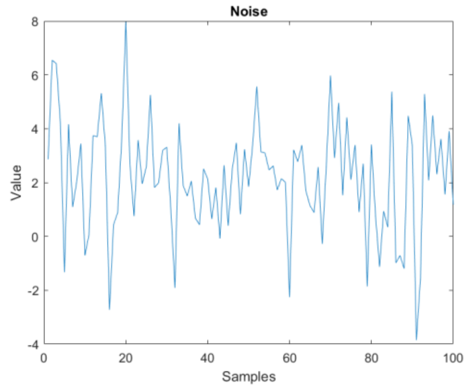
$$\sigma_{mean} = \sqrt{\mathrm{Var}(\bar{X})} = \frac{\sigma_{noise}}{\sqrt{N}}$$

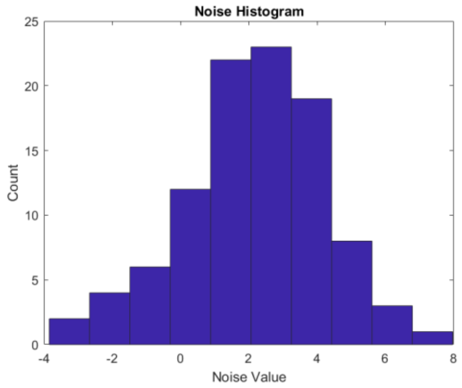
# Handling Measurement Noise

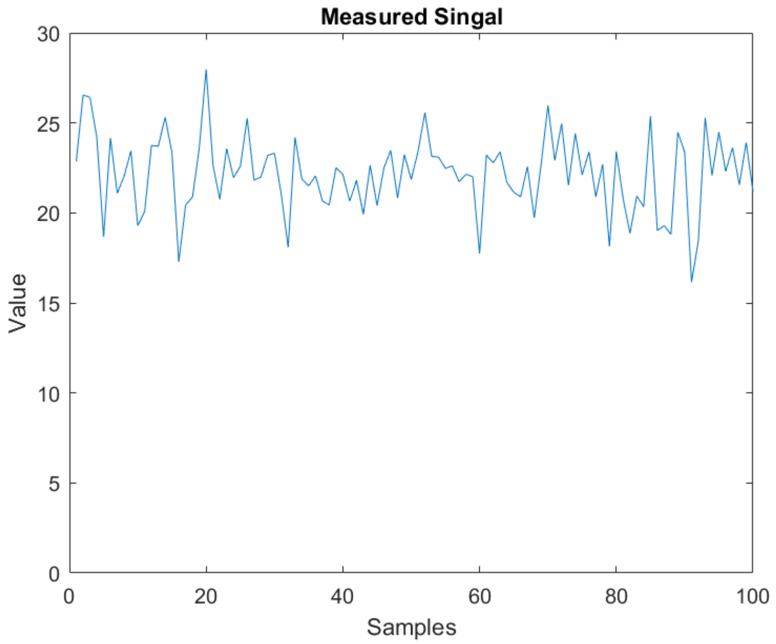
## ***Handling Noise***

- How many measurements should I take? Answer: how certain do I need to be?

$\mu_n = 2, \sigma_n = 2$

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$S = \mu_m - \mu_n = 20.1689$

$$\sigma_{mean} = \frac{\sigma_{noise}}{\sqrt{N}}$$

For a standard error of no more than **ε**:

$$N = \left(\frac{\sigma_{noise}}{\varepsilon}\right)^2$$

E.g.,:

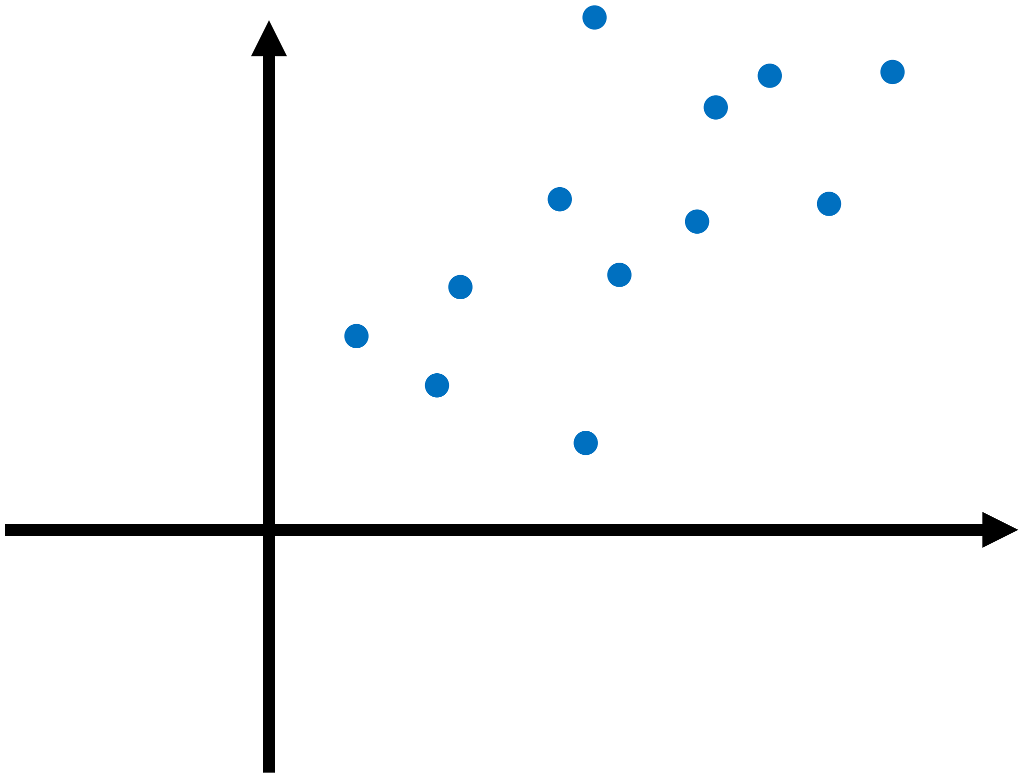
For $\varepsilon = 0.1$ (final guess to be accurate to within ±0.1)

$$N = \left(\frac{\sigma_{noise}}{\varepsilon}\right)^2 = \left(\frac{2}{0.1}\right)^2 = 400$$

# Handling Measurement Noise

***RANSAC***

- Suppose that we have the following measurements, and we know that our model is a linear function: $F(x) = ax + b$



# Handling Measurement Noise

***RANSAC:: RANdom SAmple Consensus***

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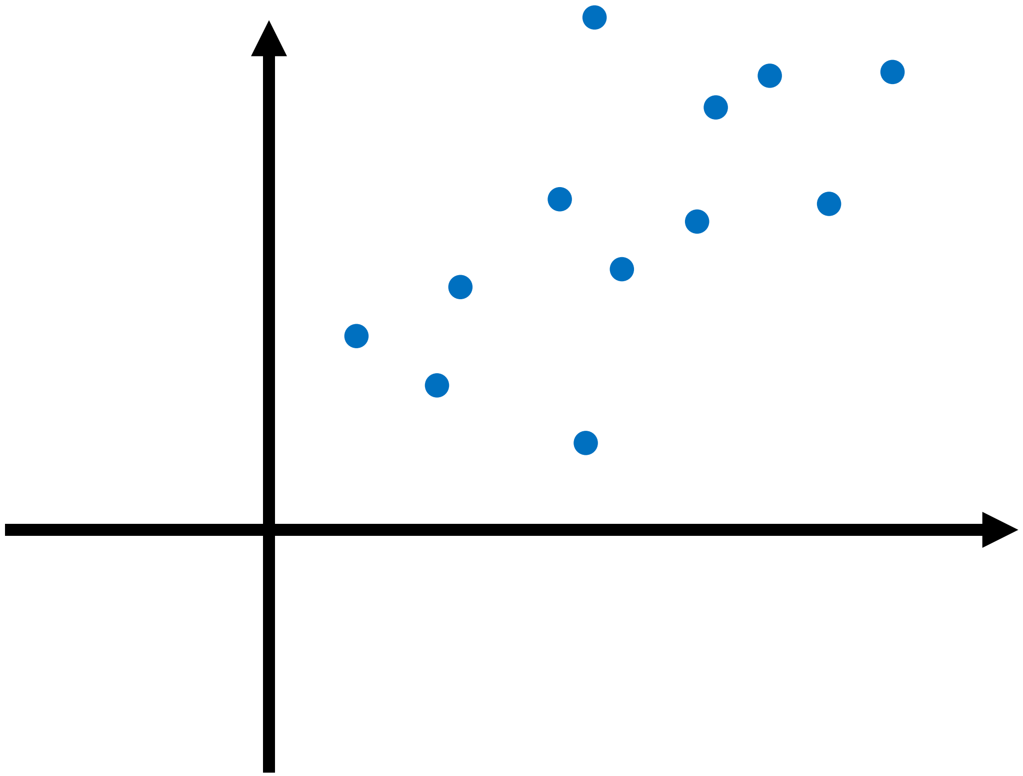

**RANSAC steps:**

1. Randomly select $n$ number of samples.
   $n$ being the degrees of freedom for our model. 2 in our example.
2. Fit our model into these samples.
3. Measure the number of inliers given a tolerance distance $d$.
   This is our assumption's score.
4. Repeat steps 1 through 3 for a different set of $n$ samples.

Our best hypothesis is the assumption with the most inliers.

*Trial and error*

# Handling Measurement Noise

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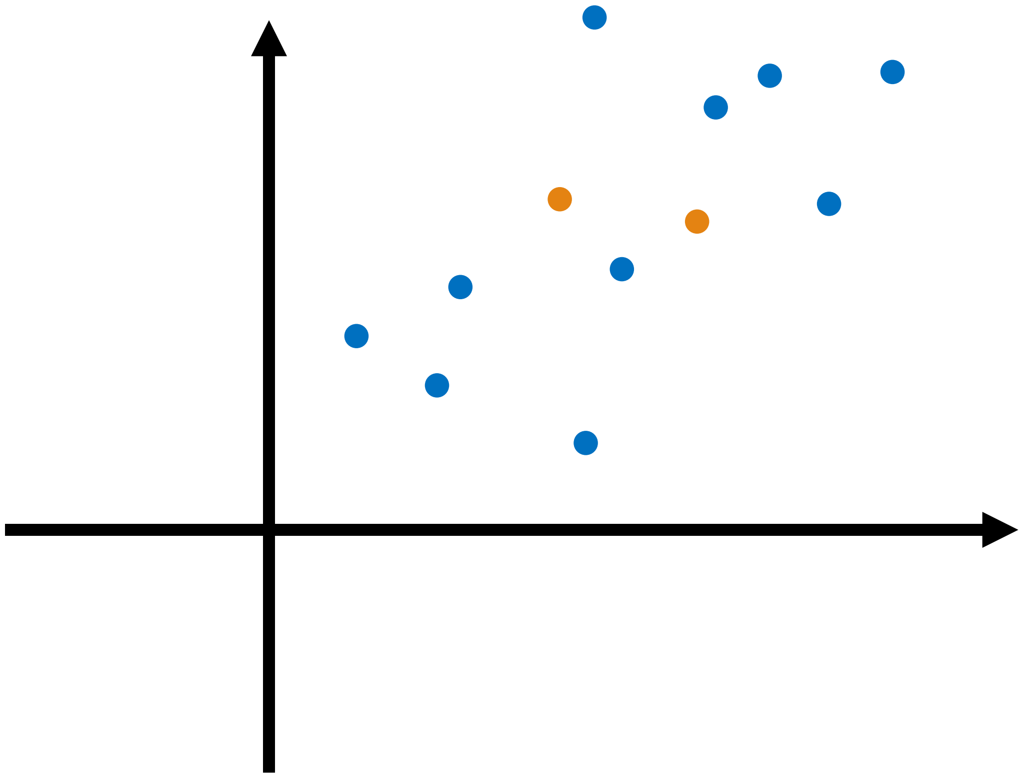

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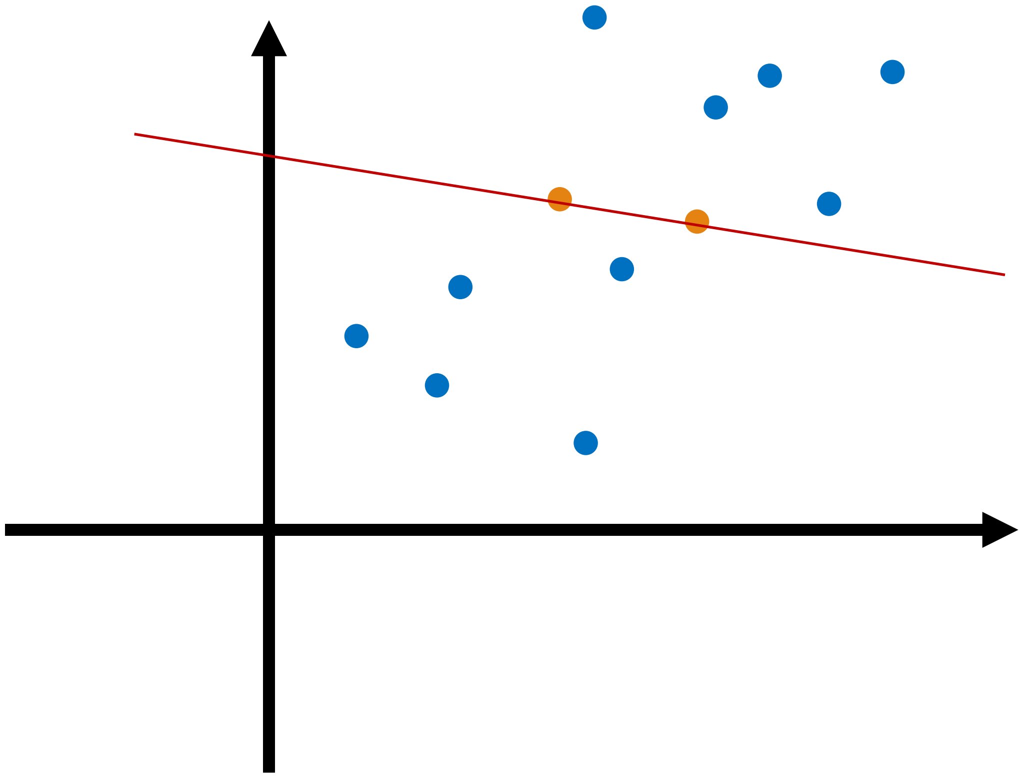

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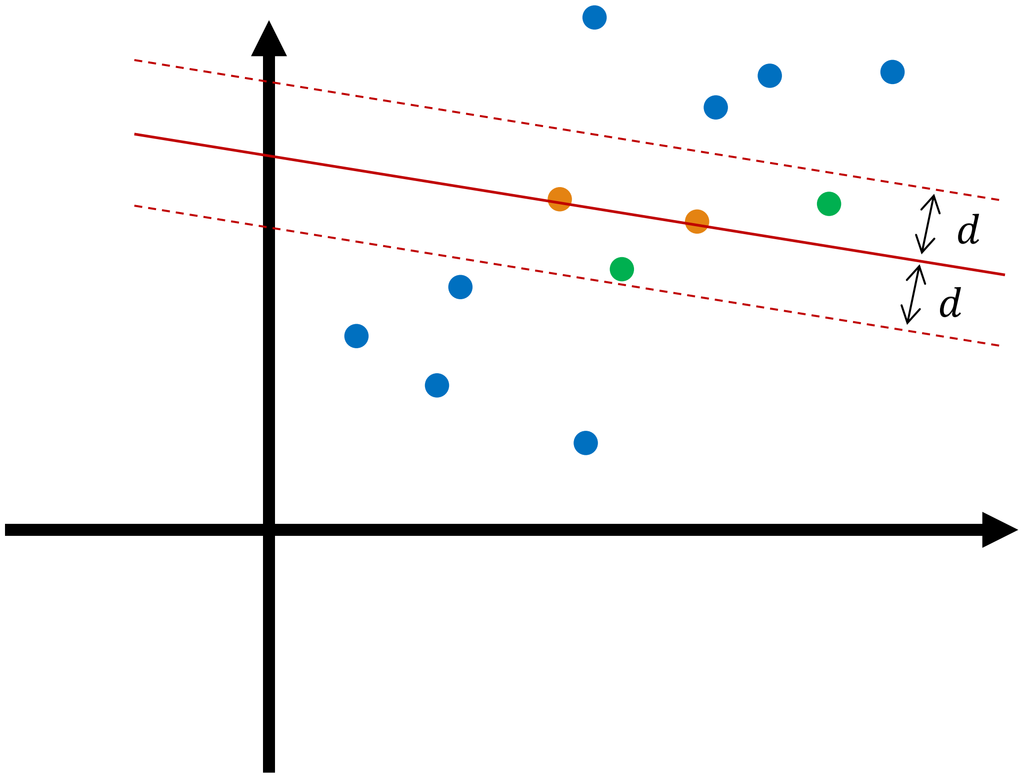


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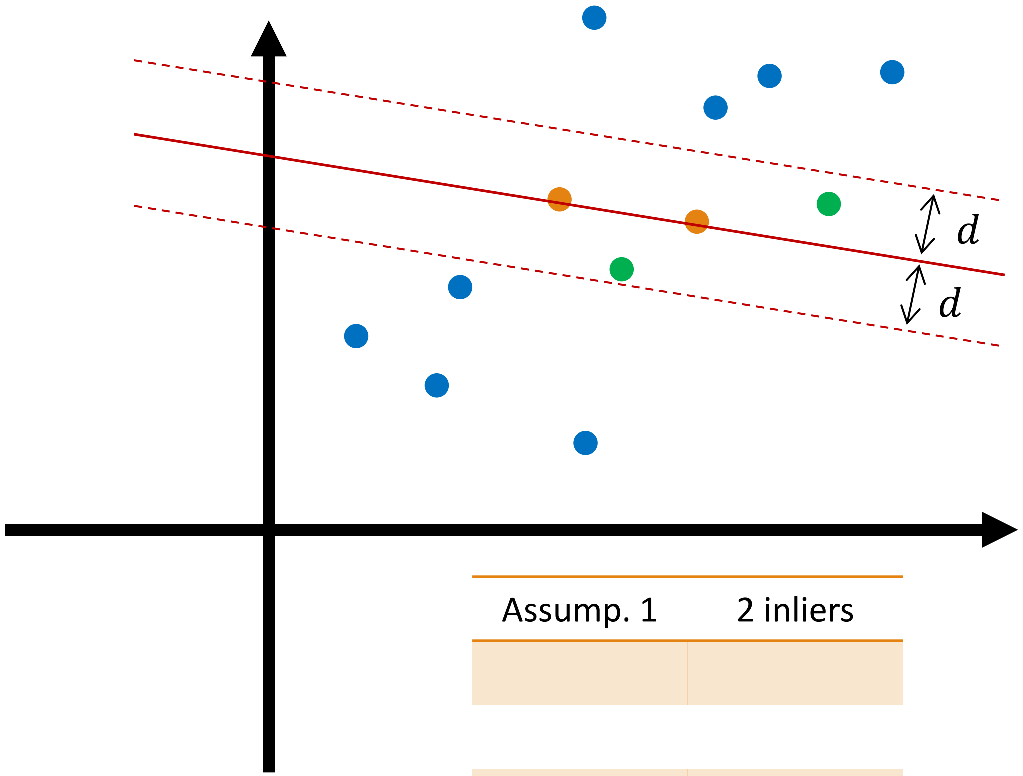


| Assump. 1 | 2 inliers |
|---|---|
| | |
| | |
| | |

**RANSAC steps:**

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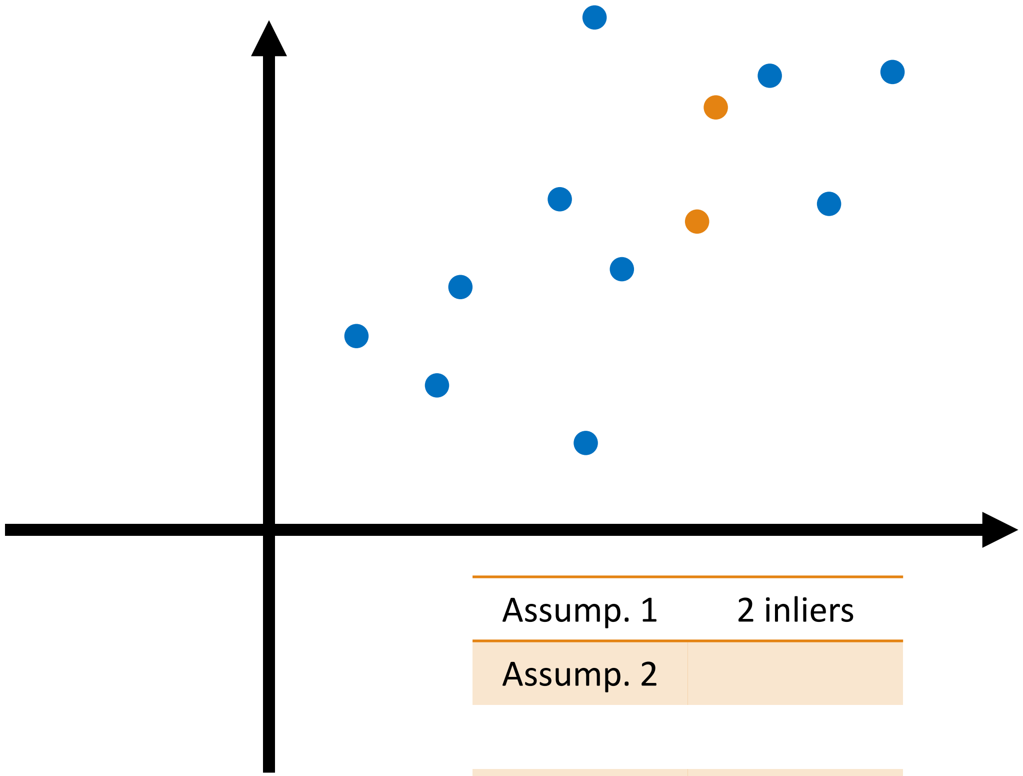

| | |
|---|---|
| Assump. 1 | 2 inliers |
| Assump. 2 | |
| | |
| | |

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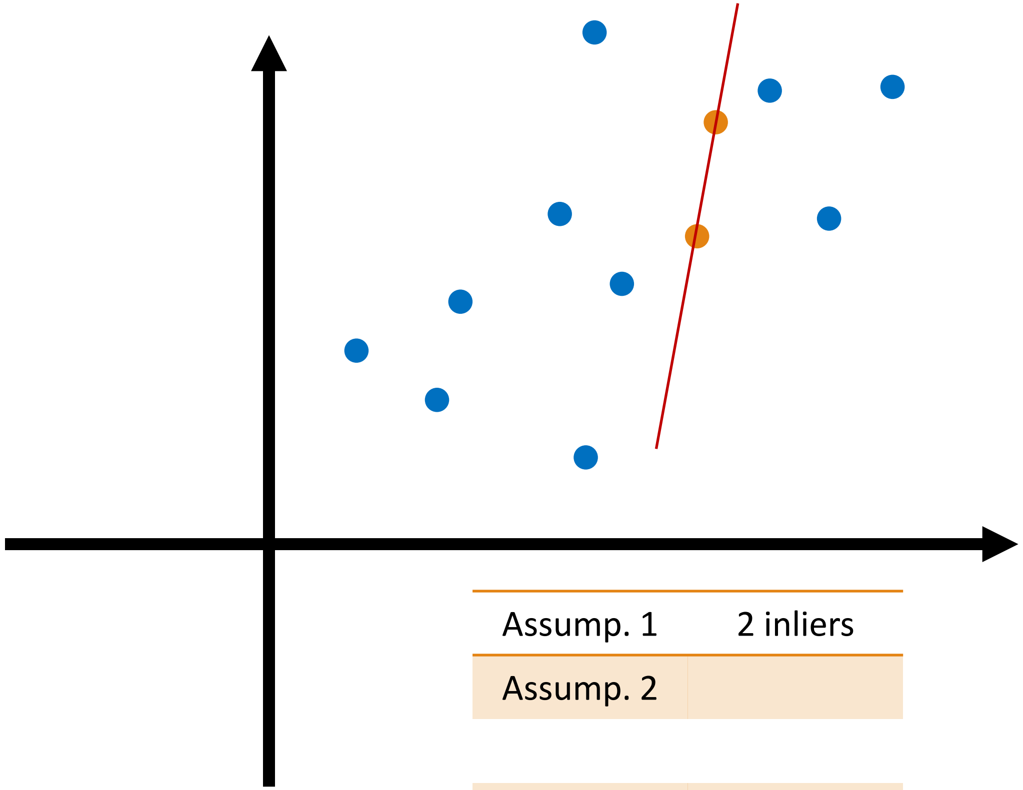

| | |
|---|---|
| Assump. 1 | 2 inliers |
| Assump. 2 | |
| | |
| | |

**RANSAC steps:**

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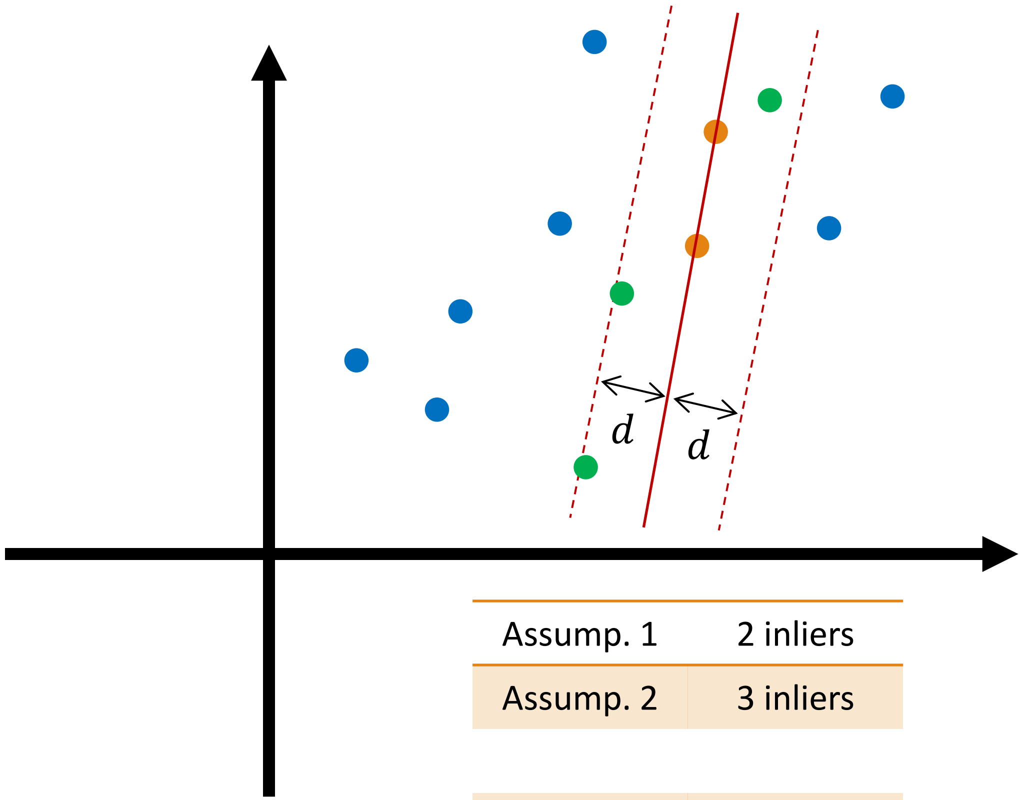


| | |
|---|---|
| Assump. 1 | 2 inliers |
| Assump. 2 | 3 inliers |
| | |
| | |

**RANSAC steps:**

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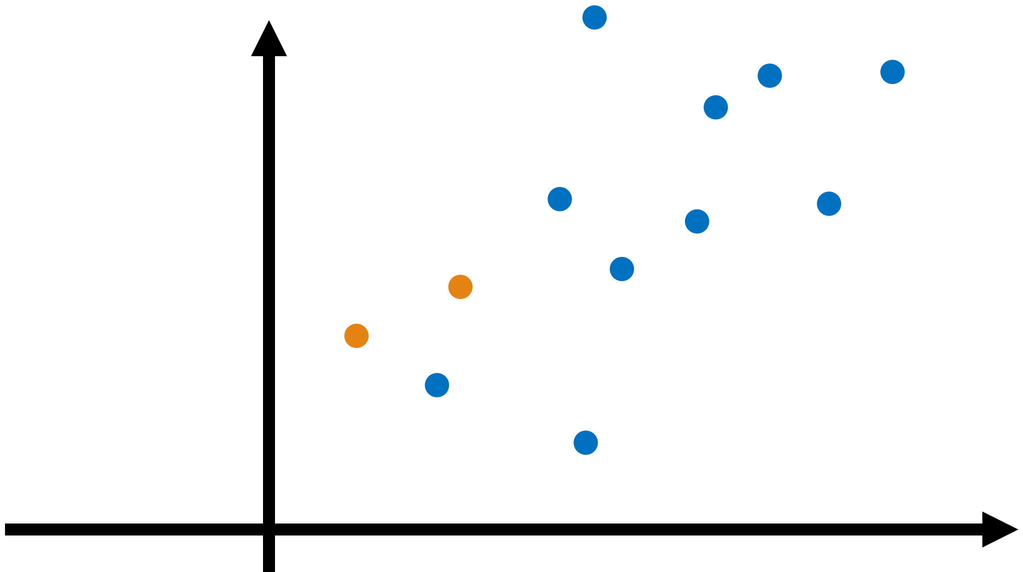
Our best hypothesis is the assumption with the most inliers.

# Handling Measurement Noise

***RANSAC:: RANdom SAmple Consensus***

- Suppose that we have the following measurements, and we know that our model is a linear function: $F(x) = ax + b$



| | |
|---|---|
| Assump. 1 | 2 inliers |
| Assump. 2 | 3 inliers |
| Assump. 3 | |
| | |

**RANSAC steps:**

1. Randomly select $n$ number of samples.
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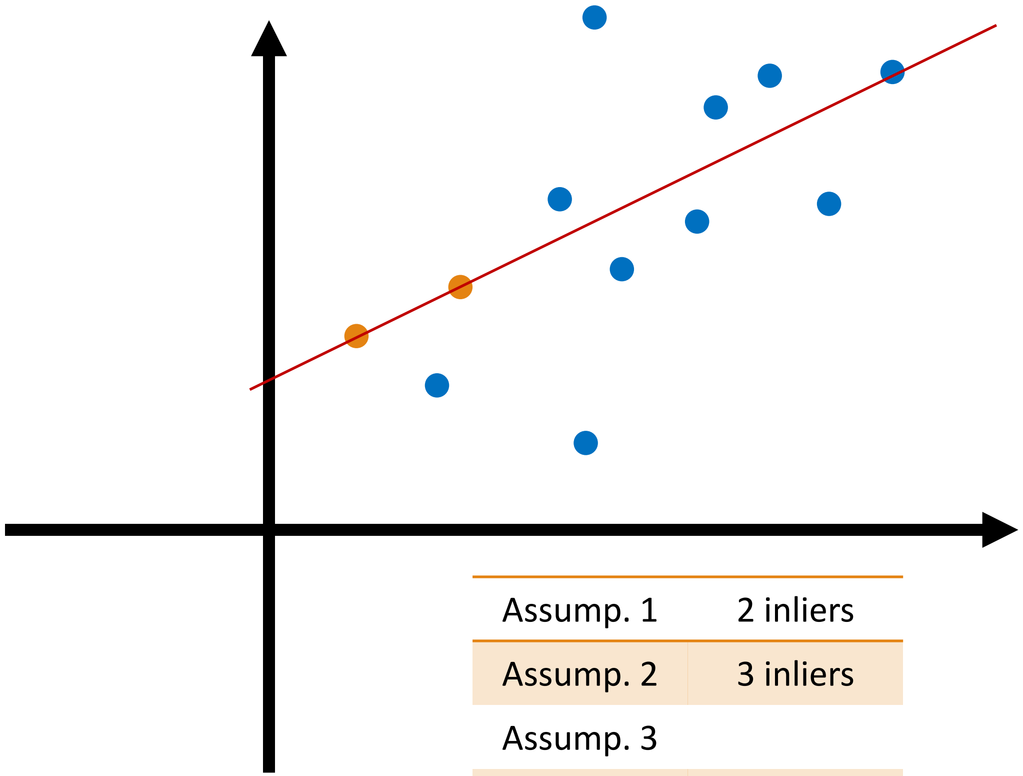

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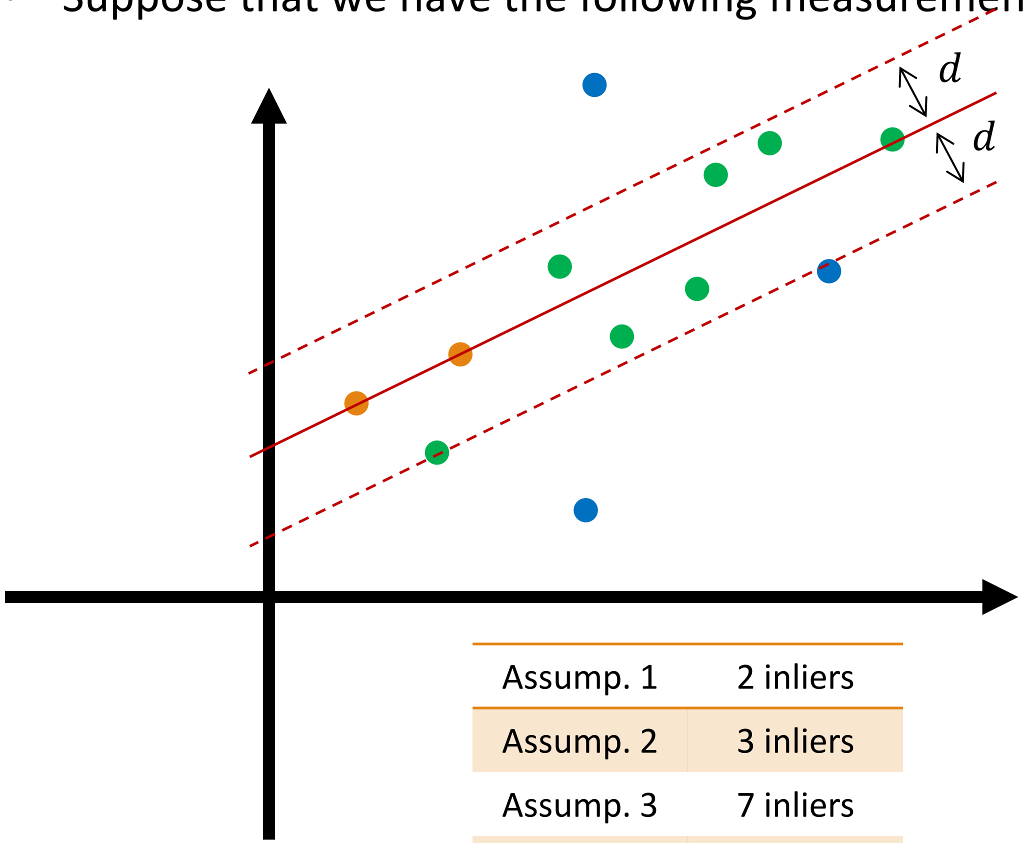


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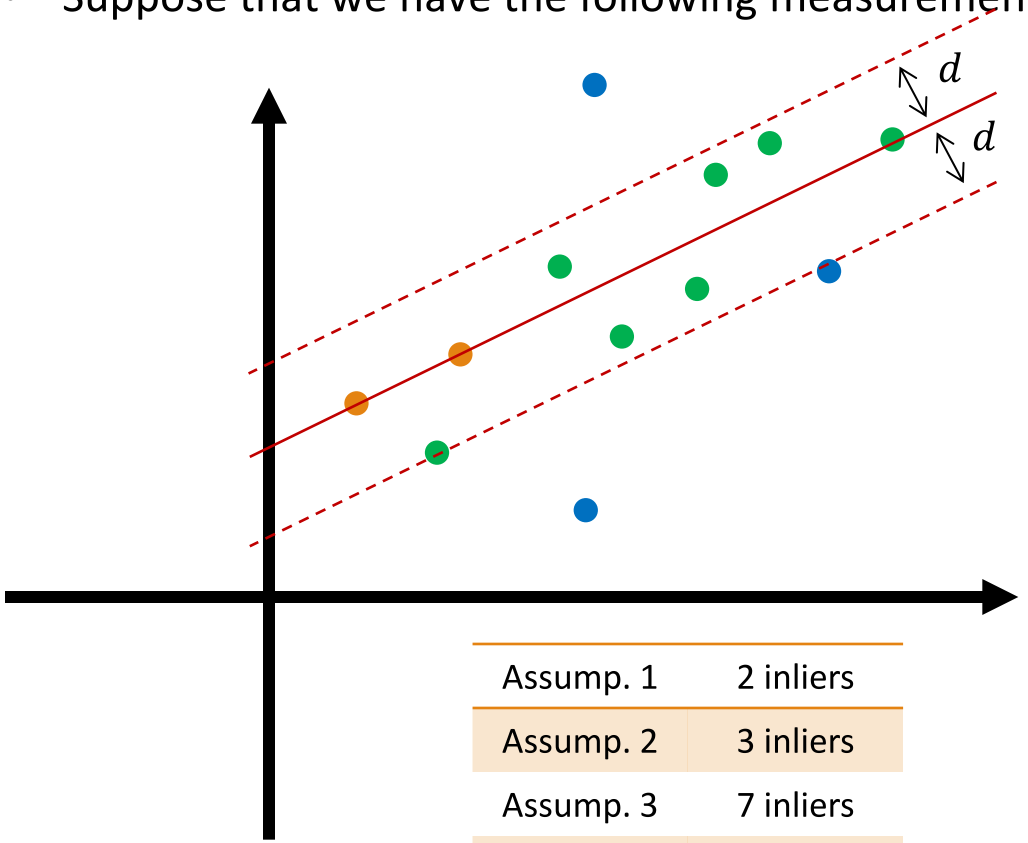


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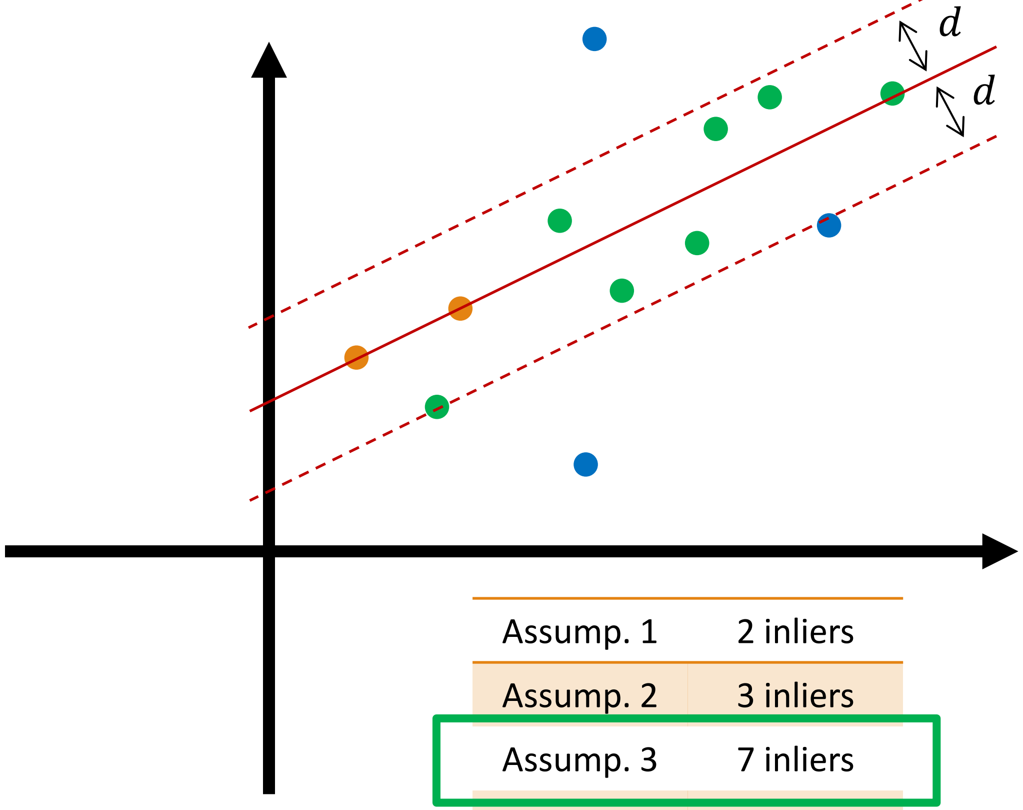


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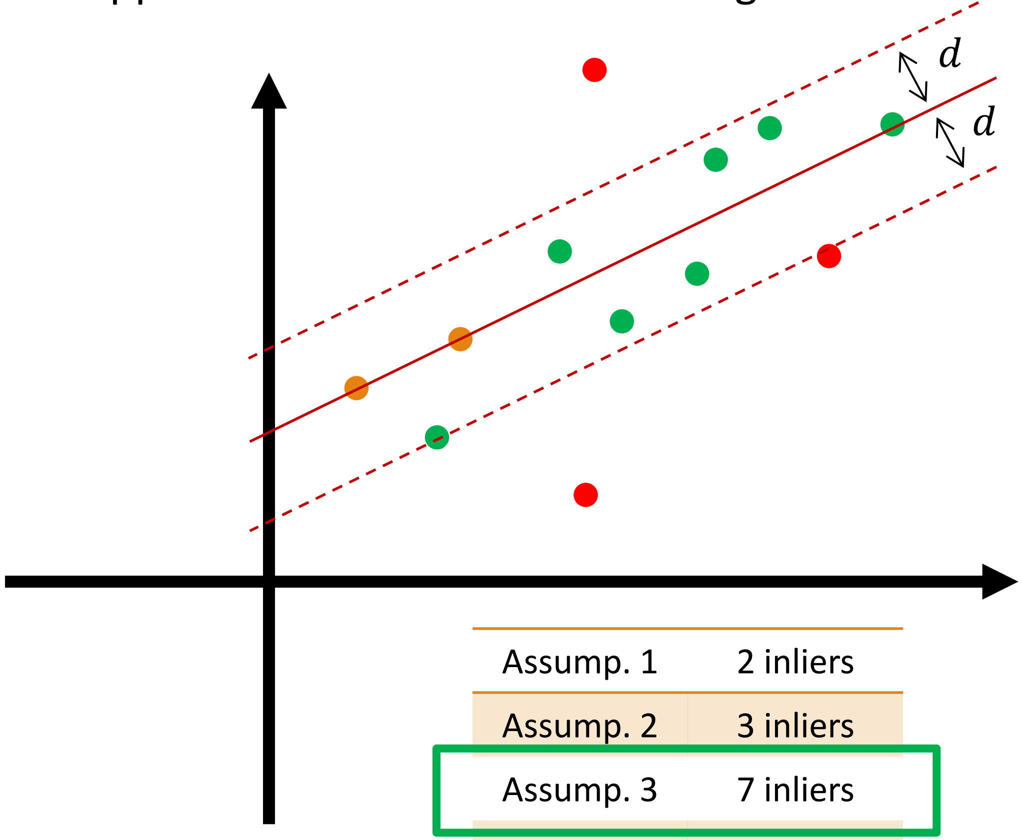


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Through RANSAC we can also identify outliers. Values that are so noisy or resulted from an external disturbance and should not be considered for our model.

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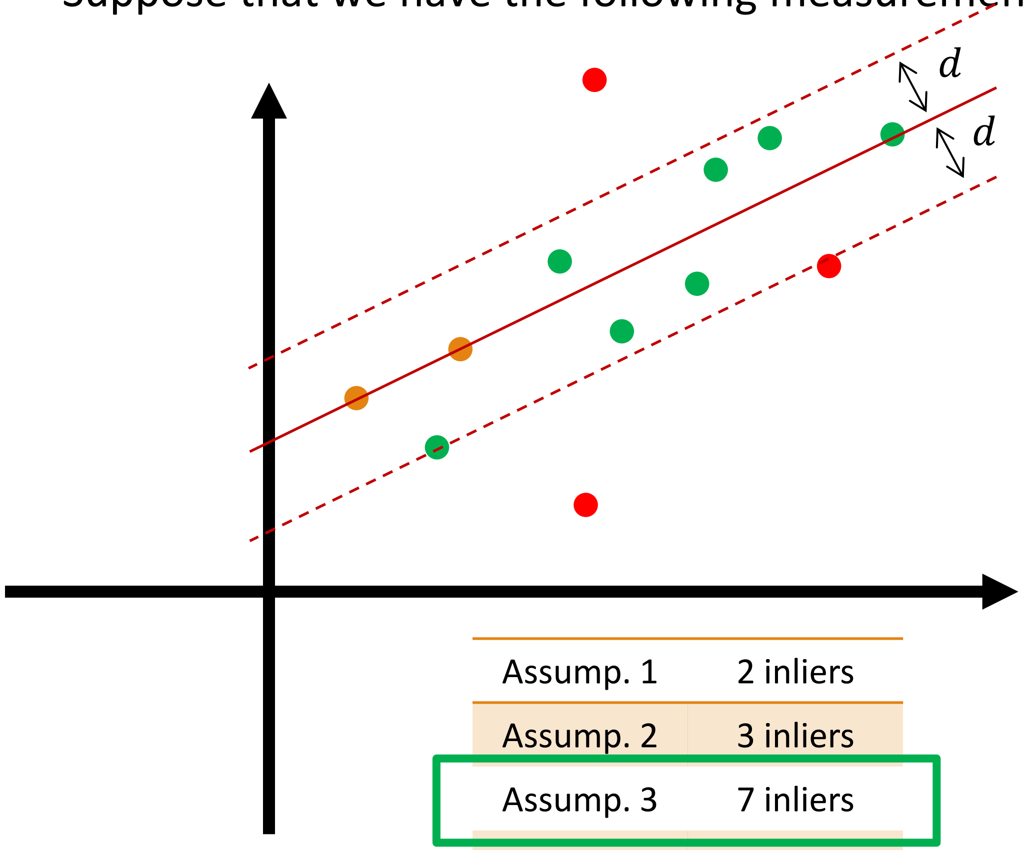


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How do we define $d$?

- Do we know the distribution of noise?
  - Find $\mu$ and $\sigma$
  - Define an acceptable noise tolerance with respect to $\sigma$
  - Set: $d = \kappa \cdot \sigma$

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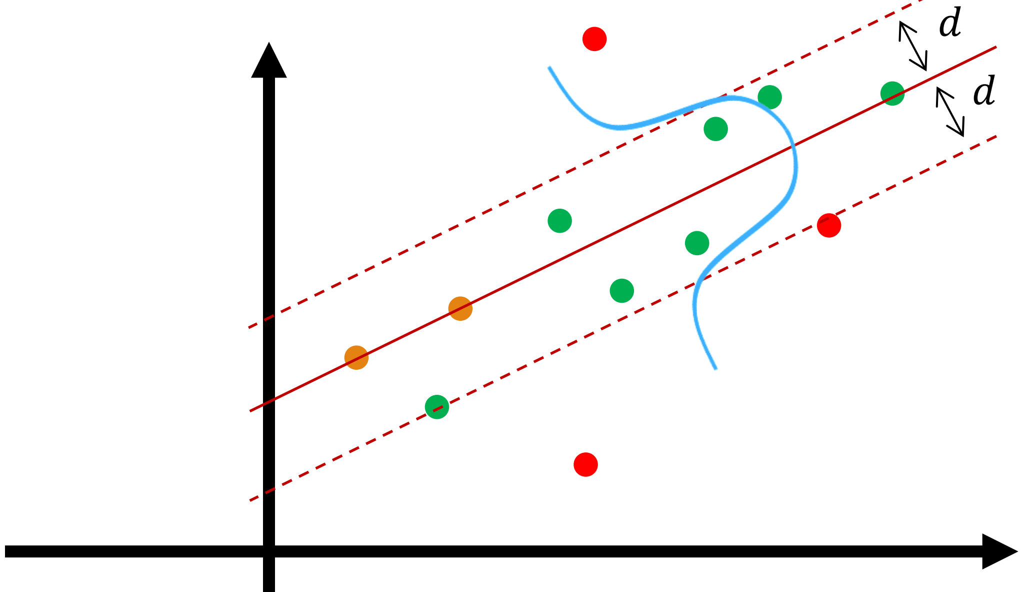


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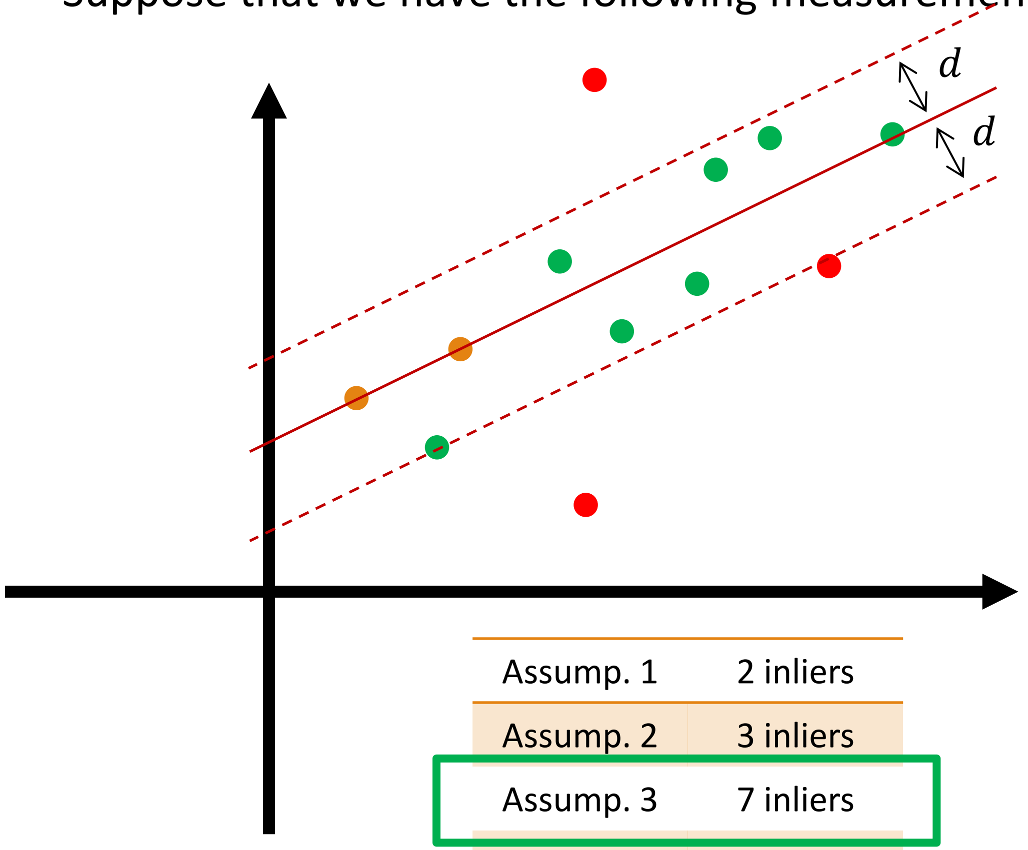


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How many iterations/assumptions to evaluate?

$$I = \frac{\log(1-p)}{\log(1-(1-e)^{dof})}$$

$p$: probability succeeding in finding a proper solution after $I$ iterations (finding a sample of inliers = $dof$)

$e$: outlier ratio (#outliers/#samples)

$dof$: degrees of freedom

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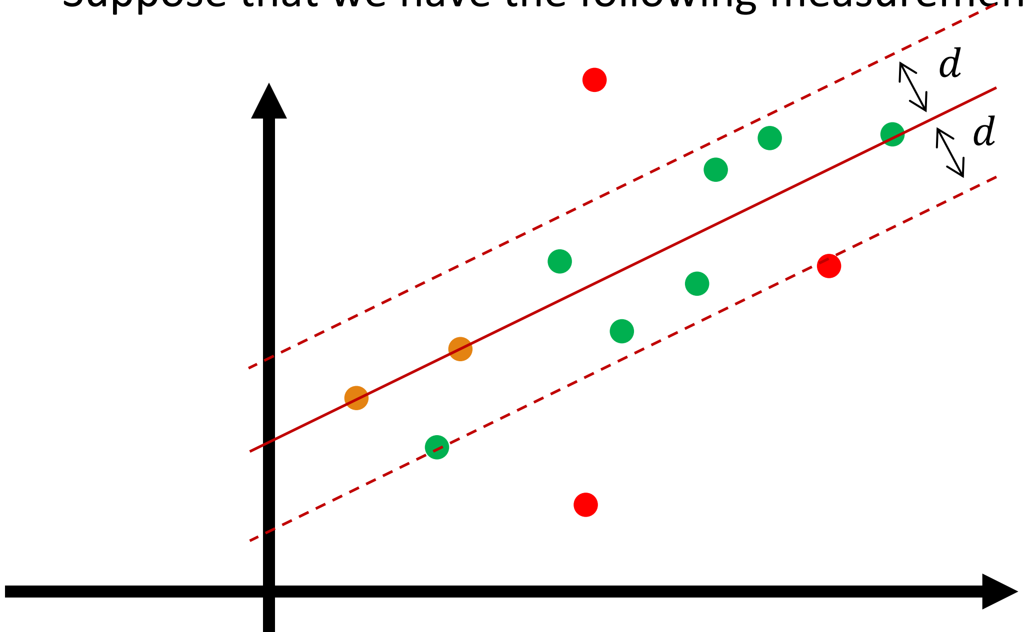


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Proof in a single sample attempt (one RANSAC iteration):

- Prob. of drawing a single inlier: $1 - e$
- Prob. of drawing as many inliers as needed: $(1-e)^{dof}$
- Prob. of not drawing only inliers: $1 - (1-e)^{dof}$
- Prob. of failing:

$$(1-q) = 1 - (1-e)^{dof}$$

Prob. of succeeding ($q$)

Prob. of failing ($1-q$)

# Handling Measurement Noise

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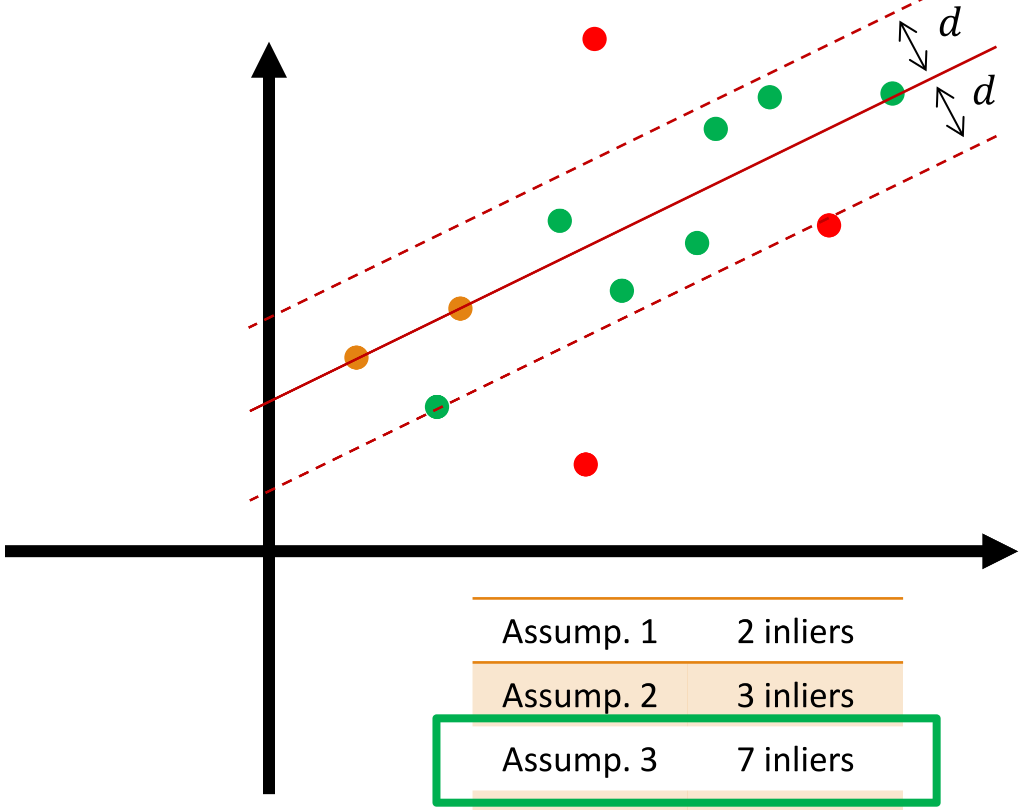


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Prob. of failing in a single sample attempt (one RANSAC iteration):

$$(1-q) = 1-(1-e)^{dof}$$

Prob. of failing in $I$ sample attempts ($T$ RANSAC iteration):

$$(1-p) = \left(1-(1-e)^{dof}\right)^{I}$$

Prob. of succeeding ($p$)

# Handling Measurement Noise

***<u>RANSAC:: RANdom SAmple Consensus</u>***

- Any kind of model