QE12. Measurement Theory

- Principles of Quantum Measurements

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Lesson #3

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In maths…

To define that representation we need a further class of operators: **projection operators** or **projectors** for short. The projector **P**, onto the eigenstate $|a_i>$ is defined by

$$\mathbf{P}_i := |a_i\rangle\langle a_i|$$

Application of $\mathbf{P}_i$ to an arbitrary state $|\psi_i>$ yields a multiple of $|a_i>$

$$\mathbf{P}_i|\psi\rangle = |a_i\rangle\langle a_i|\psi\rangle = \langle a_i|\psi\rangle|a_i\rangle$$

where $|<a_i|\psi>|$ the "length" of the projection of $|\psi_i>$ onto the unit vector $|a_i>$. And if $<a_i|a_j> = \delta_{ij}$ then

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j;\ \text{especially}\ \mathbf{P}_i^2 = \mathbf{P}_i$$

As the $\mathbf{P}_i$ cover "all directions" of Hilbert space we obtain a completeness relation:

$$\sum_{i=1}^{d}\mathbf{P}_i = \sum_{i=1}^{d}|a_i\rangle\langle a_i| = \mathbf{1}$$

Trace of a matrix A : Tr(A) the sum of the diagonal elements

$\mathbf{P}_i$ is Hermitian: $\mathbf{P}_i = \mathbf{P}_i^\dagger$

Projection postulate

Assume a quantum system prepared in a state $|\psi>$ and a single measurement of the observable $\mathbf{A}$ is performed. This cycle of preparation and measurement is repeated many times so that the notion of probability used in the postulate makes sense.
Or imagine an ensemble containing a large number of independent copies of the quantum system, all prepared in the same state $|\psi>$. $\mathbf{A}$ is measured for all system copies independently.

<u>Projection postulate</u> : A single measurement of the observable $\mathbf{A}$ in the normalized state $|\psi>$ yields one of the eigenvalues a_i of $\mathbf{A}$ with probability $|<a_i|\psi>|^2$. Immediately after the measurement the system is in the (normalized) state

$$\frac{\mathbf{P}_i\ |\psi\rangle}{||\mathbf{P}_i\ |\psi\rangle||}$$

where $\mathbf{P}_i$ is the projection operator onto the subspace of eigenstates of $\mathbf{A}$ with eigenvalue a_i .

Projective measurement result

In general it is not possible to predict the outcome of a single measurement. A measurement of $\mathbf{A}$ on an ensemble of systems as discussed above yields the *average* (expectation value)

$$\langle \mathbf{A} \rangle := \langle \psi | \mathbf{A} | \psi \rangle$$

with deviations described by the **variance** (the square of the standard deviation): $\langle (\mathbf{A} - \langle \mathbf{A} \rangle)^2 \rangle \geq 0$

The probability of obtaining outcome i for a given state $|\psi>$

$$p_i = < \psi | \mathbf{P}_i | \psi >$$

And the post-measurement state is given by

$$|\psi_i^{post}\rangle = \frac{\mathbf{P}_i |\psi>}{\sqrt{< \psi | \mathbf{P}_i | \psi >}}$$

In quantum mechanics the measurement change the state of a quantum system which is probabilistic and irreversible process. The observation process is irreversible.

Projectors: a methodology for a single qubit

1. Identify the projectors for the basis: $P_i = |v_i\rangle\langle v_i|$
2. Compute each projected vector: $|\psi_{P_i}\rangle = P_i\,|\psi\rangle$
3. Compute the squared magnitude of each projection: as the probability of seeing the i-th normalized projection. $\||\psi_{P_i}\rangle|^2$
4. Compute each normalized projection (each potential outcome): $|\psi_{N_i}\rangle = \dfrac{|\psi_{P_i}\rangle}{\||\psi_{P_i}\rangle|}$

This gives us each possible outcome and its associated probability:

Outcome $|\psi_{N_i}\rangle$ occurs with probability $\||\psi_{P_i}\rangle|^2$

Example 1:

Suppose we measure the state $|0>$ in the basis {|+>, |->}.

|0> = 1/√2 (|+> + |->)

Step 1: Identify the projectors for the basis.

$$
\begin{aligned}
P_+ &= |+\rangle\langle+| \\
P_- &= |-\rangle\langle-|
\end{aligned}
$$

Step 2: Compute each projected vector:

$$
\begin{aligned}
P_+|0\rangle &= |+\rangle\langle+|\,|0\rangle &= |+\rangle\langle+|\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) &= \frac{1}{\sqrt{2}}|+\rangle \\
P_-|0\rangle &= |-\rangle\langle-|\,|0\rangle &= |-\rangle\langle-|\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) &= \frac{1}{\sqrt{2}}|-\rangle
\end{aligned}
$$

Example 1:

Step 3: Compute the squared magnitude of each projection:

$$|P_+ |0\rangle|^2 = \left|\frac{1}{\sqrt{2}}|+\rangle\right|^2 = \left\langle \frac{1}{\sqrt{2}}\langle+| \,\middle|\, \frac{1}{\sqrt{2}}|+\rangle \right\rangle = \frac{1}{2}$$

$$|P_- |0\rangle|^2 = \left|\frac{1}{\sqrt{2}}|-\rangle\right|^2 = \left\langle \frac{1}{\sqrt{2}}\langle-| \,\middle|\, \frac{1}{\sqrt{2}}|-\rangle \right\rangle = \frac{1}{2}$$

Step 4: Compute each normalized projection (each potential outcome):

$$\frac{\frac{1}{\sqrt{2}}|+\rangle}{\left|\frac{1}{\sqrt{2}}|+\rangle\right|} = |+\rangle \qquad \frac{\frac{1}{\sqrt{2}}|-\rangle}{\left|\frac{1}{\sqrt{2}}|-\rangle\right|} = |-\rangle$$

This gives us possible outcomes and their associated probabilities:

$$\text{outcome } |+\rangle \text{ occurs with probability } \frac{1}{2}$$

$$\text{outcome } |-\rangle \text{ occurs with probability } \frac{1}{2}$$

Example: two-qubits states

Consider the two-qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

And suppose we use the two-qubit standard basis for two-qubit measurement: $|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle$

Step 1: Identify the measurement basis projectors:

$$
\begin{aligned}
P_{00} &= |00\rangle\langle 00| \\
P_{01} &= |01\rangle\langle 01| \\
P_{10} &= |10\rangle\langle 10| \\
P_{11} &= |11\rangle\langle 11|
\end{aligned}
$$

For 2 qubits in the standard basis, the shorthand is:

$$
\begin{aligned}
|0\rangle \otimes |0\rangle &= |00\rangle \\
|0\rangle \otimes |1\rangle &= |01\rangle \\
|1\rangle \otimes |0\rangle &= |10\rangle \\
|1\rangle \otimes |1\rangle &= |11\rangle
\end{aligned}
$$

Example: two-qubits states

Step 2: Use the above to compute projected vectors:

$$P_{00}\,|\psi\rangle = |00\rangle\langle 00|\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = 0$$

$$P_{01}\,|\psi\rangle = |01\rangle\langle 01|\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}|01\rangle$$

$$P_{10}\,|\psi\rangle = |10\rangle\langle 10|\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = \frac{1}{\sqrt{2}}|10\rangle$$

$$P_{11}\,|\psi\rangle = |11\rangle\langle 11|\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right) = 0$$

Step 3: The magnitudes of (non-zero) projected vectors are their probabilities:

$$|P_{01}\,|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}|01\rangle\right|^2 = \frac{1}{2}$$

$$|P_{10}\,|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}|10\rangle\right|^2 = \frac{1}{2}$$

Step 4: Normalize the projections to get the outcome vectors:

$$\frac{1}{|P_{01}\,|\psi\rangle|}P_{01}\,|\psi\rangle = |01\rangle$$

$$\frac{1}{|P_{10}\,|\psi\rangle|}P_{10}\,|\psi\rangle = |10\rangle$$

Tip for calculations

Notice the usefulness of Dirac notation, over matrices, in the second step, for example:

$$
\begin{aligned}
&|01\rangle\langle 01| \left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \right) \\
&= \left(\boxed{\frac{1}{\sqrt{2}}}|01\rangle \boxed{\langle 01|\,|01\rangle} + \frac{1}{\sqrt{2}}|01\rangle \boxed{\langle 01|\,|10\rangle} \right) \\
&= \frac{1}{\sqrt{2}}|01\rangle
\end{aligned}
$$

Scalar movement

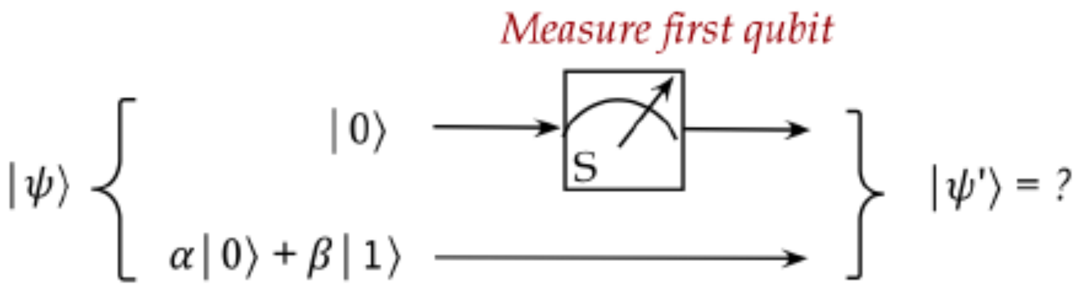
inner-product = 1

inner-product = 0

Example #3

Suppose you have a 2-qubit state and we want to measure only the first qubit. The input two-qubit vector is $|\psi\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$

After measuring only the first qubit, what are the possible states, and with what probabilities?



Solution:
Clearly, since the first (top) qubit is already |0>, measuring this qubit alone should leave the top qubit as |0>. Intuition suggests that because there's no entanglement, the second qubit should be the same. That is, after measurement, that state should be

$$|\psi'\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \quad \text{Or} \quad |\psi'\rangle = \alpha(|0\rangle \otimes |0\rangle) + \beta(|0\rangle \otimes |1\rangle) = \alpha|00\rangle + \beta|01\rangle$$

Projective measurement: extending the theory to multiple qubits

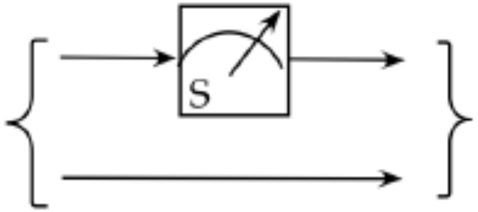
We want to extend the theory to be able to handle different measurement scenarios with qubits.

Examples of scenarios include:

- Measuring all qubits simultaneously.
- Measuring just one qubit amongst the qubits.
- Measuring a subset of qubits from the qubits.
- And each case, having the freedom to use a variety of measurement bases.
- There are three aspects to extending single-qubit projective measurement to multiple qubits:
- Understanding how the particular measurement splits the whole -qubit space into orthogonal subspaces.
- Building -qubit projectors accordingly.
- Seeing if it helps to construct the -qubit projectors from smaller projectors (such as 1-qubit projectors).

Measure first qubit in S-basis

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$



What are outcomes and their probabilities?

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

We want to measure the one qubit.
We will go about addressing this question in several stages:

- We'll first examine the potential vectors that result from subjecting the 2-qubit vector to 1-qubit measurement.
- These potential vectors will form a vector space.
- Then, we'll build the projectors for these spaces.
- We'll also see that the projectors can be built from tensoring.

Stage 1: if you measure the first qubit, what are the possible 2-qubit vectors that result?
Example: first qubit results in |0>.
Second qubit could potentially be any state.
Describe this as the space

$$V_1 = \text{span}\{\,|00\rangle\,,|01\rangle\,\}$$

Any vector in this space has the first qubit as |0> .

Similarly, the space of 2-qubit vectors that correspond to "first qubit is |1>" is

$$V_2 = \text{span}\{\,|10\rangle\,,|11\rangle\,\}$$

Stage 2: Thus, the potential results of first-qubit measurement lie in two orthogonal subspaces:

$$V = V_1 \cup V_2$$

where

$$\begin{aligned} V_1 &= \text{span}\{\,|00\rangle\,,|01\rangle\,\} && \text{First qubit } |0\rangle \\ V_2 &= \text{span}\{\,|10\rangle\,,|11\rangle\,\} && \text{First qubit } |1\rangle \end{aligned}$$

Stage 3: Now define projectors for each of these subspaces

$$\begin{aligned} P_{V_1} &= |00\rangle\langle 00| + |01\rangle\langle 01| \\ P_{V_2} &= |10\rangle\langle 10| + |11\rangle\langle 11| \end{aligned}$$

Note: applying in Dirac form makes it easy to see why these are projectors for those subspaces:

- Consider any vector

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

- Then

$$\begin{aligned} P_{V_1}|\psi\rangle &= \text{projection of } |\psi\rangle \text{ on } V_1 \\ &= (|00\rangle\langle 00| + |01\rangle\langle 01|)(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) \\ &= \alpha|00\rangle + \beta|01\rangle \\ &= \text{A vector in } V_1 \end{aligned}$$

Recall: $V_1 = \text{span}\{|00\rangle, |01\rangle\}$

- Similarly,

$$\begin{aligned} P_{V_2}|\psi\rangle &= \text{projection of } |\psi\rangle \text{ on } V_2 \\ &= \gamma|10\rangle + \delta|11\rangle \\ &= \text{A vector in } V_2 \end{aligned}$$

The *outcomes* of measurement and their probabilities are

$$\begin{array}{ll} \text{normalized } P_{V_1}|\psi\rangle & \text{occurs with probability } |P_{V_1}|\psi\rangle|^2 \\ \text{normalized } P_{V_2}|\psi\rangle & \text{occurs with probability } |P_{V_2}|\psi\rangle|^2 \end{array}$$

- Squared magnitudes of the projections are:

$$\begin{aligned} |P_{V_1}|\psi\rangle|^2 &= |\alpha|^2 + |\beta|^2 \\ |P_{V_2}|\psi\rangle|^2 &= |\gamma|^2 + |\delta|^2 \end{aligned}$$

- Normalized projections are:

$$\begin{aligned} \frac{1}{|P_{V_1}|\psi\rangle|} P_{V_1}|\psi\rangle &= \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} (\alpha|00\rangle + \beta|01\rangle) \\ \frac{1}{|P_{V_2}|\psi\rangle|} P_{V_2}|\psi\rangle &= \frac{1}{\sqrt{|\gamma|^2 + |\delta|^2}} (\gamma|10\rangle + \delta|11\rangle) \end{aligned}$$

Let's take a closer look at the first projected vector to see that it makes sense:

- Recall we started in state

$$|\psi\rangle \;=\; \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

- Let's write this as

$$|\psi\rangle \;=\; |0\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle\,) \;+\; |1\rangle \otimes (\gamma\,|0\rangle + \delta\,|1\rangle\,)$$

  Here, we've just separated out the first qubit for emphasis.
- If we measure the first qubit as $|0\rangle$, then the second should be "untouched" in state $\alpha\,|0\rangle + \beta\,|1\rangle$.
- And the resulting 2-qubit vector should be

$$|0\rangle \otimes (\alpha\,|0\rangle + \beta\,|1\rangle\,) \;=\; \alpha\,|00\rangle + \beta\,|01\rangle$$

- But the latter is not normalized, and so, the normalized vector is:

$$\frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}}(\alpha\,|00\rangle + \beta\,|01\rangle\,)$$

Can the 2-qubit projectors be built out of smaller 1-qubit projectors?

Recall that the two 1-qubit S-basis projectors are

$$
\begin{aligned}
P_0 &= |0\rangle\langle 0| \\
P_1 &= |1\rangle\langle 1|
\end{aligned}
$$

Since we're not measuring the 2nd-qubit, the only projector that keeps the qubit the same is the identity

$$I = |0\rangle\langle 0| + |1\rangle\langle 1|$$

(in Dirac form).

Thus, one can construct the 2-qubit projectors via tensoring

$$
\begin{aligned}
P_{V_1} &= P_0 \otimes I &= |0\rangle\langle 0| \otimes (\,|0\rangle\langle 0| + |1\rangle\langle 1|\,) \\
P_{V_2} &= P_1 \otimes I &= |1\rangle\langle 1| \otimes (\,|0\rangle\langle 0| + |1\rangle\langle 1|\,)
\end{aligned}
$$

Let's work out the first one to see the details:

$$
\begin{aligned}
P_0 \otimes I &= |0\rangle\langle 0| \otimes (\, |0\rangle\langle 0| + |1\rangle\langle 1|\,) && \text{Tensor of projectors} \\
&= (\, |0\rangle\langle 0| \otimes |0\rangle\langle 0|\,) + (\, |0\rangle\langle 0| \otimes |1\rangle\langle 1|\,) && \text{Tensor properties} \\
&= (\, |0 \otimes 0\rangle\langle 0 \otimes 0|\,) + (\, |0 \otimes 1\rangle\langle 0 \otimes 1|\,) && \text{Proposition 4.5} \\
&= |00\rangle\langle 00| + |01\rangle\langle 01| && \text{Shorthand notation}
\end{aligned}
$$

Please prove that:

$$
|v\rangle\langle v| \otimes |w\rangle\langle w| = |v \otimes w\rangle\langle v \otimes w|
$$

Finally, let's remind ourselves of alternative tensoring notation:

- We can write

$$|v\rangle\langle v| \otimes |w\rangle\langle w| \quad = \quad |\mathbf{v}\rangle\, |\mathbf{w}\rangle\, \langle \mathbf{v}|\, \langle \mathbf{w}| \qquad (\text{ Also written as } |v, w\rangle \langle v, w|)$$

- Thus, we could also have written the earlier projector example as:

$$\begin{aligned} P_0 \otimes I &= |0\rangle\langle 0| \otimes (\, |0\rangle\langle 0| + |1\rangle\langle 1|\,) \\ &= (\, |0\rangle\langle 0| \otimes |0\rangle\langle 0|\,) + (\, |0\rangle\langle 0| \otimes |1\rangle\langle 1|\,) \\ &= (\, |\mathbf{0}\rangle\, |\mathbf{0}\rangle\, \langle \mathbf{0}|\, \langle \mathbf{0}|\,) + (\, |\mathbf{0}\rangle\, |\mathbf{1}\rangle\, \langle \mathbf{0}|\, \langle \mathbf{1}|\,) \\ &= |00\rangle\langle 00| + |01\rangle\langle 01| \end{aligned}$$

- By convention, we do *not* write $|00\rangle$ as $|0, 0\rangle$.

Homework

Consider the single-qubit projectors $P_+ = |+\rangle\langle+|$ and $P_- = |-\rangle\langle-|$.

1. Show that $(P_+ \otimes I)\,|00\rangle = \frac{1}{\sqrt{2}}|+,0\rangle$ and $(P_+ \otimes I)\,|11\rangle = \frac{1}{\sqrt{2}}|+,1\rangle$
2. Show that $\frac{1}{\sqrt{2}}|+,0\rangle + \frac{1}{\sqrt{2}}|+,1\rangle = |+,+\rangle$
3. Expand $(P_+ \otimes I)$ into a matrix and use that to show $(P_+ \otimes I)\,|\psi\rangle = |+,+\rangle$ (normalized) where $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

End of Lesson