QE12. Measurement Theory (L2)

- Classic Measurement Theory
- Principles of Quantum Measurements

Dr Panagiotis Dimitrakis

DEMOCRITUS UNIVERSITY OF THRACE | DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING

MSc in QUANTUM COMPUTING AND QUANTUM TECNNOLOGIES

NATIONAL CENTRE FOR SCIENTIFIC RESEARCH "DEMOKRITOS"

In maths…

To define that representation we need a further class of operators: **projection operators** or **projectors** for short. The projector $\mathbf{P}_i$ onto the eigenstate $|a_i>$ is defined by

$$\mathbf{P}_i := |a_i\rangle\langle a_i|$$

Application of $\mathbf{P}_i$ to an arbitrary state $|\psi_i>$ yields a multiple of $|a_i>$

$$\mathbf{P}_i|\psi\rangle = |a_i\rangle\langle a_i|\psi\rangle = \langle a_i|\psi\rangle|a_i\rangle$$

where $|<a_i|\psi>|$ the “length” of the projection of $|\psi_i>$ onto the unit vector $|a_i>$. And if $<a_i|a_j> = \delta_{ij}$ then

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j; \text{ especially } \mathbf{P}_i^2 = \mathbf{P}_i$$

As the $\mathbf{P}_i$ cover “all directions” of Hilbert space we obtain a completeness relation:

$$\sum_{i=1}^{d}\mathbf{P}_i = \sum_{i=1}^{d}|a_i\rangle\langle a_i| = \mathbf{1}$$

$\mathbf{P}_i$ is Hermitian: $\mathbf{P}_i = \mathbf{P}_i^\dagger$

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Projection postulate

Assume a quantum system prepared in a state $|\psi>$ and a single measurement of the observable $\mathbf{A}$ is performed. This cycle of preparation and measurement is repeated many times so that the notion of probability used in the postulate makes sense.
Or imagine an ensemble containing a large number of independent copies of the quantum system, all prepared in the same state $|\psi>$. $\mathbf{A}$ is measured for all system copies independently.

<u>Projection postulate</u> : A single measurement of the observable $\mathbf{A}$ in the normalized state $|\psi>$ yields one of the eigenvalues a_i of $\mathbf{A}$ with probability $|<a_i|\psi>|^2$. Immediately after the measurement the system is in the (normalized) state

$$\frac{\mathbf{P}_i\ |\psi\rangle}{||\mathbf{P}_i\ |\psi\rangle||}$$

where $\mathbf{P}_i$ is the projection operator onto the subspace of eigenstates of $\mathbf{A}$ with eigenvalue a_i .

Projective measurement result

In general it is not possible to predict the outcome of a single measurement. A measurement of $\mathbf{A}$ on an ensemble of systems as discussed above yields the *average* (expectation value)

$$\langle \mathbf{A} \rangle := \langle \psi | \mathbf{A} | \psi \rangle$$

with deviations described by the **variance** (the square of the standard deviation): $\langle (\mathbf{A} - \langle \mathbf{A} \rangle)^2 \rangle \geq 0$

The probability of obtaining outcome i for a given state $|\psi\rangle$

$$p_i = \langle \psi | \mathbf{P}_i | \psi \rangle \quad or \quad p_i = \langle \psi | P_i^\dagger P_i | \psi \rangle$$

And the post-measurement state is given by

$$|\psi_i^{post}\rangle = \frac{\mathbf{P}_i |\psi\rangle}{\sqrt{\langle \psi | \mathbf{P}_i | \psi \rangle}}$$

In quantum mechanics the measurement change the state of a quantum system which is probabilistic and irreversible process. The observation process is irreversible.

Example

Projective measurement is performed on a qubit state ψ: $|\psi>=\alpha|0>+\beta|1>$ using a projector $\mathbf{P}$. Find the probability the qubit state is (a) |0> and (b) |1>. What is the post-measurement state of |0>, |1> respectively?

Solution:

We know that $p_i = <\psi|\mathbf{P}_i|\psi>$ thus

$p_0 = <\psi|\mathbf{P}_0|\psi> = <\psi|0><0|\psi> = a^*a = |a|^2$

$p_1 = <\psi|\mathbf{P}_1|\psi> = <\psi|1><1|\psi> = \beta^*\beta = |\beta|^2$

$\mathbf{P}_0 = |0><0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathbf{P}_1 = |1><1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Post measurement states are

$$\frac{P_0|\psi>}{\sqrt{p_0}} = \frac{|0\rangle\langle 0|\psi\rangle}{\sqrt{|a|^2}} = \frac{a}{|a|}|0\rangle = |\psi_0>$$

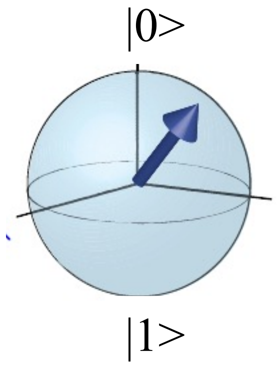
$$\frac{P_1|\psi>}{\sqrt{p_1}} = \frac{|1\rangle\langle 1|\psi\rangle}{\sqrt{|\beta|^2}} = \frac{\beta}{|\beta|}|1\rangle = |\psi_1>$$

Bloch sphere interpretation

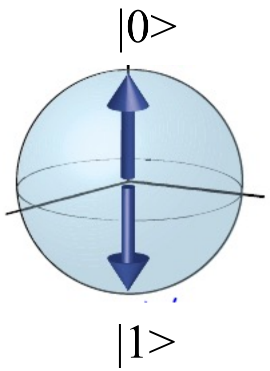
Measurement Basis is $\{|0>, |1>\}$

A quantum state is described by

$$|\psi> = \alpha|0> + \beta|1>$$



After a projective measurement is completed the qubit will be in either one of its computational basis states.
In a repeated measurement the projected state will be measured with certainty.



Information content in a single qubit state

- infinite number of qubit states
- but single measurement reveals only 0 or 1 with probabilities $|\alpha|^2$ or $|\beta|^2$
- measurement will collapse state vector on basis state
- to determine α and β an infinite number of measurements has to be made

Multiple Qubits : Two qubits case

2 Classical Bits

Bit 1	Bit 1
0	0
0	1
1	0
1	1

- 2^n different states (here n=2)
- but only one is realized at any given time

2 Qubits with quantum states

$\lvert 00\rangle$
$\lvert 01\rangle$
$\lvert 10\rangle$
$\lvert 11\rangle$

- 2^n basis states (n=2)
- can be realized simultaneously
- quantum parallelism

2^n complex coefficients describe quantum state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Normalization condition

$$\sum_{i,j} \alpha_{i,j} = 1$$

Composite Quantum Systems

QM postulate: The state space of a composite systems is the tensor product of the state spaces of the component physical systems. If the component systems have states ψ_i the composite system state is

$$|\psi> = |\psi_1> \otimes |\psi_2> \otimes \ldots \otimes |\psi_i>$$

Example:

$|\psi_1> = \alpha_1|0> + \beta_1|1>$

$|\psi_2> = \alpha_2|0> + \beta_2|1>$

Thus

$$|\psi> = |\psi_1> \otimes |\psi_2> = |\psi_1\psi_2> = \alpha_1\alpha_2|00> + \alpha_1\beta_2|01> + \beta_1\alpha_2|10> + \beta_1\beta_2|11>$$
$$= \alpha_{00}|00> + \alpha_{01}|01> + \alpha_{10}|10> + \alpha_{11}|11>$$

Information content in multiple qubits

- 2^n complex coefficients describe the state of a composite quantum system with n qubits
- Imagine to have 500 qubits, then 2^{500} complex coefficients describe their state.
- How to store this state?
  - ✓ 2^{500} is larger than the number of atoms in the universe.
  - ✓ It is impossible in classical bits.
  - ✓ This is also why it is hard to simulate quantum systems on classical computers.
- A quantum computer would be much more efficient than a classical computer at simulating quantum systems.

- Make use of the information that can be stored in qubits for quantum information processing!

Entanglement

Definition: An entangled state of a composite system is a state that cannot be written as a product state of the component systems.

E.g.: an entangled 2-qubit state (one of the Bell states)
$|\psi\rangle = 1/\sqrt{2}\ (|00\rangle + |11\rangle)$

What is special about this state? Try to write it as a product state!

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle\ ,\ |\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$|\psi_1\psi_2\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

$$|\psi\rangle = |\psi_1\psi_2\rangle \Rightarrow \alpha_1\alpha_2 = 1/\sqrt{2} \text{ and } \beta_1\beta_2 = 1/\sqrt{2}$$

$$\Rightarrow \alpha_1\beta_2 \neq 0 \text{ and } \beta_1\alpha_2 \neq 0\ !!!$$

It is not possible! This state is special, it is entangled! Use this property as a resource in quantum information processing:

- super dense coding
- teleportation
- error correction

Measurement of a single qubit in an entangled state

$|\psi\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$

Measurement of state "0" of the first qubit "1":

$$p_1(0) = \langle\psi|\mathbf{P}_1 \otimes \mathbf{I}|\psi\rangle = 1/\sqrt{2}\,\langle 00|\,1/\sqrt{2}\,|00\rangle = 1/2$$

Post measurement state:

$$|\psi_0\rangle = \frac{P_1 \otimes I|\psi\rangle}{\sqrt{p_1(0)}} = \frac{1/\sqrt{2}\,|\psi\rangle}{\sqrt{1/2}} = |00\rangle$$

Measurement of qubit two given that the first qubit was measured at state |00> will then result with certainty in the same result:

$$p_2(0) = \langle\psi_0|\mathbf{I}\otimes\mathbf{P}_2|\psi_0\rangle = 1$$

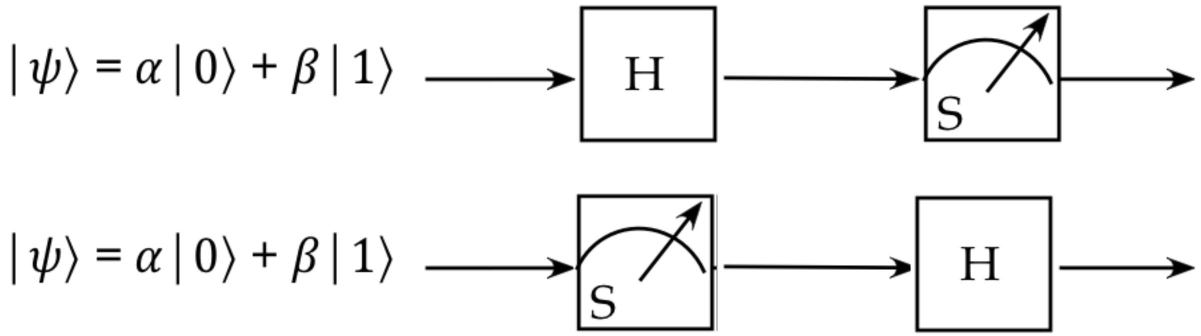
$\mathbf{P}_A \otimes \mathbf{I}$ **:** *measure an observable which acts on A only and leaves B unaffected*

The two measurement results are correlated!

- Correlations in quantum systems can be stronger than correlations in classical systems.
- This can be generally proven using the Bell inequalities which will be discussed later.
- Make use of such correlations as a resource for information processing (teleportation, error correction etc)

Homework

Consider the two circuits below, each given the same input.



1. Write down the possible states of the outputs.
2. Calculate the probabilities associated with each output state.
3. Replace Hadamard gate with another one and repeat step 1 and 2.

End of Lesson 2