Load, Stress and Strain

Collapse of the Tacoma Narrows bridge in 1940.

Fundamentals of Machine Elements, 3rd ed. Schmid, Hamrock and Jacobson.

Design Procedure 1: Critical Section and Loading

To establish the critical section and the critical loading, the designer:

1. Considers the external loads applied to a machine (e.g., a gyroscope)
2. Considers the external loads applied to an element within the machine (e.g., a ball bearing)
3. Locates the critical section within the machine element (e.g., the inner race)
4. Determines the loading at the critical section (e.g., contact stresses)

Example 1: Simple Crane

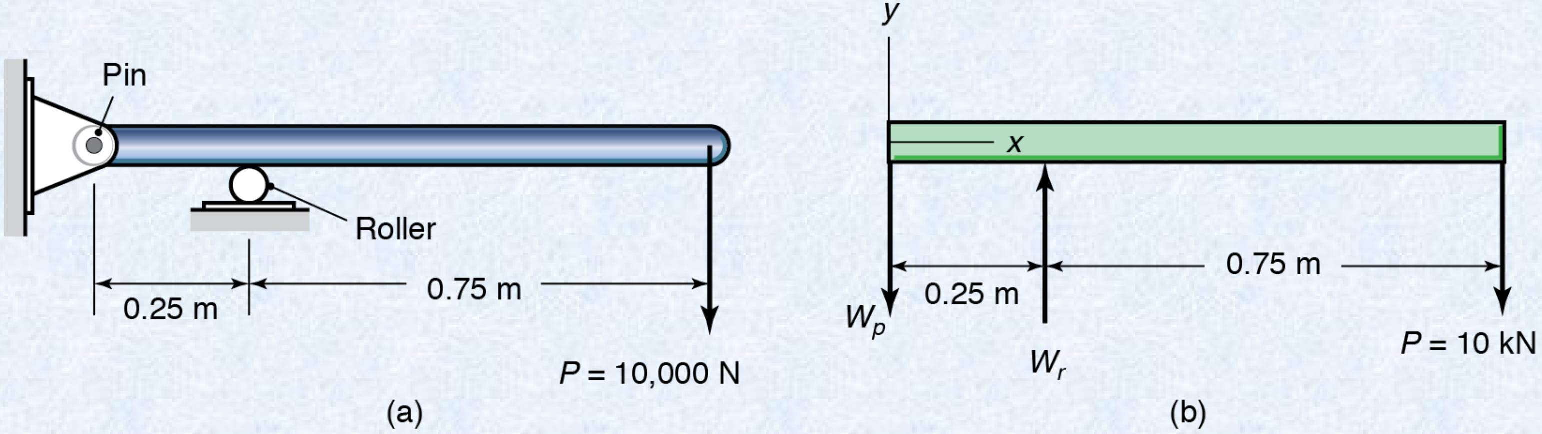


Figure 1: A schematic of a simple crane and applied forces considered in Example 1. (a) Assembly drawing; (b) free-body diagram of forces acting on the beam.

Load Classification

Any applied load can be classified with respect to time in the following ways:

1. *Static load* - Load is gradually applied and equilibrium is reached in a relatively short time. The structure experiences no dynamic effects.
2. *Sustained load* - Load, such as the weight of a structure, is constant over a long time.
3. *Impact load* - Load is rapidly applied. An impact load is usually attributed to an energy imparted to a system.
4. *Cyclic load* - Load can vary and even reverse its direction and has a characteristic period with respect to time.

The load can also be classified with respect to the area over which it is applied:

1. *Concentrated load* - Load is applied to an area much smaller than the loaded member.
2. *Distributed load* - Load is spread along a large area. An example would be the weight of books on a bookshelf.

Load Classification

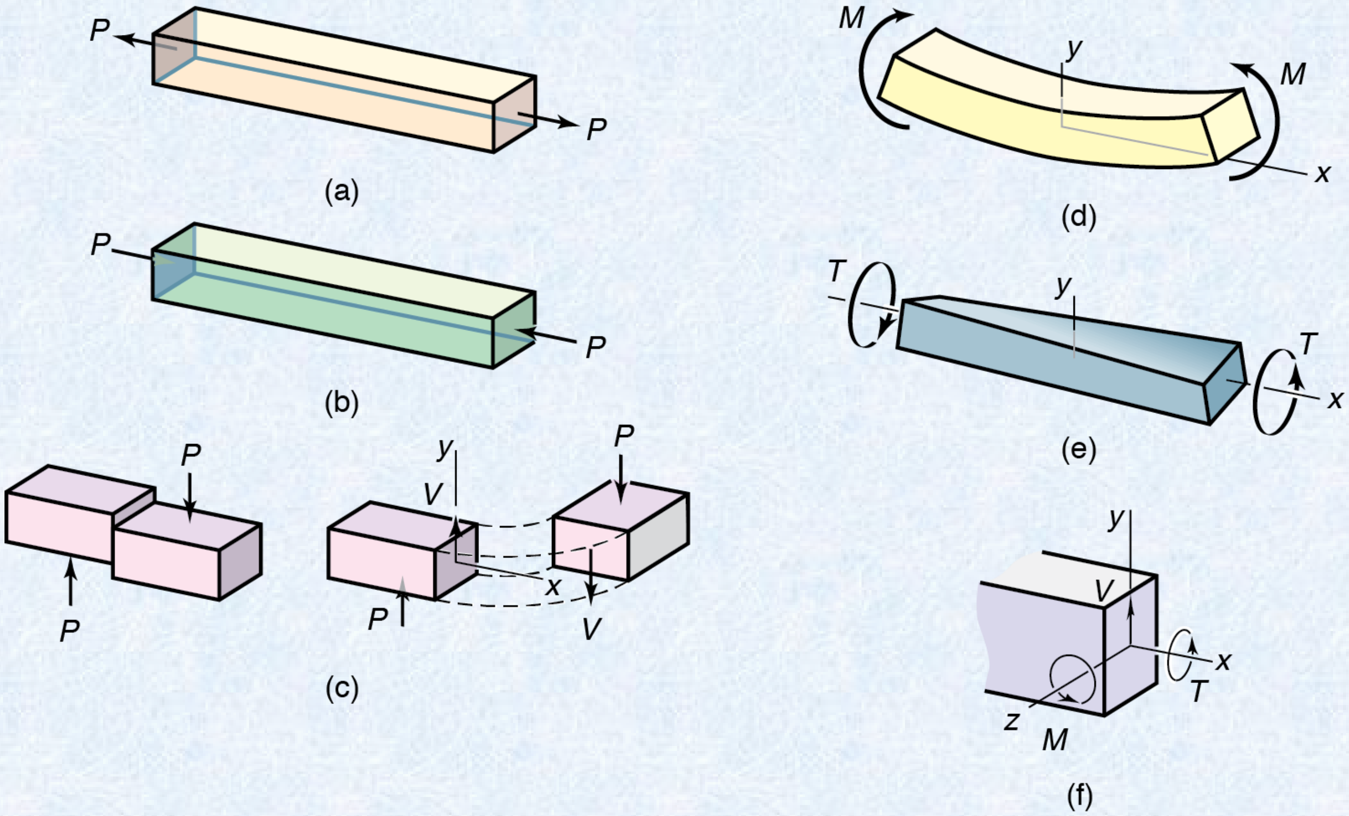


Figure 2: Load classified as to location and method of application. (a) Normal, tensile; (b) normal, compressive; (c) shear; (d) bending; (e) torsion; (f) combined.

Sign Conventions

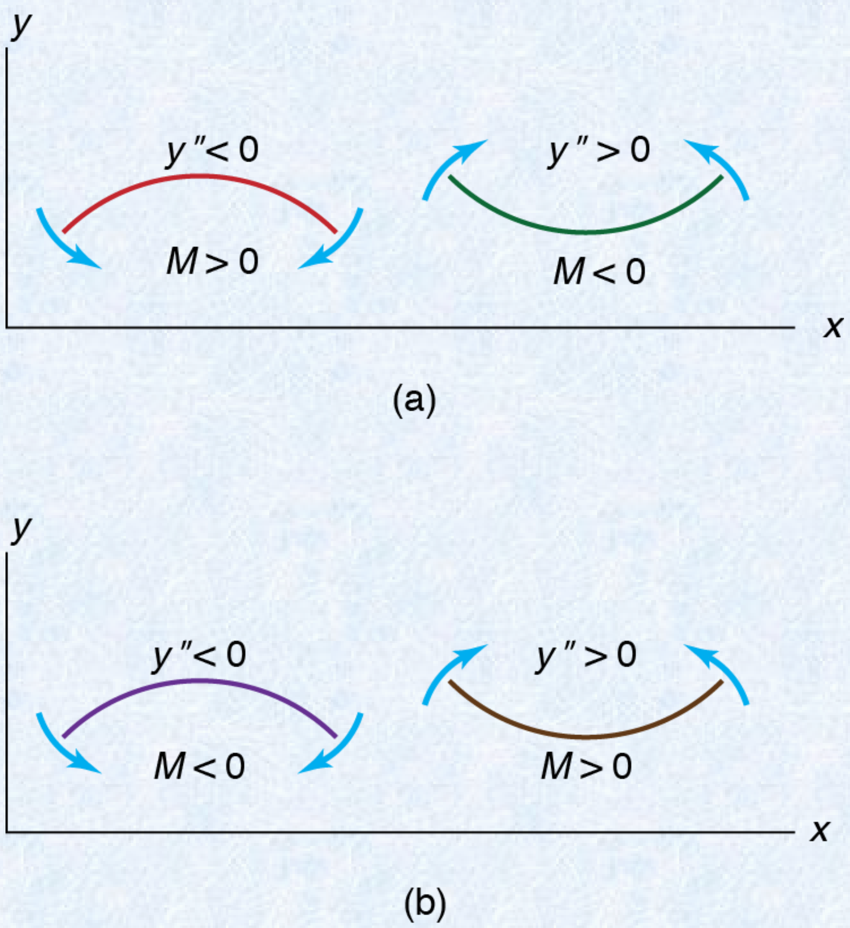


Figure 3: Sign conventions used in bending. (a) Positive moment leads to a tensile stress in the positive *y*-direction; (b) positive moment acts in a positive direction on a positive face. The sign convention shown in (b) will be used in this book.

Supports and Reactions

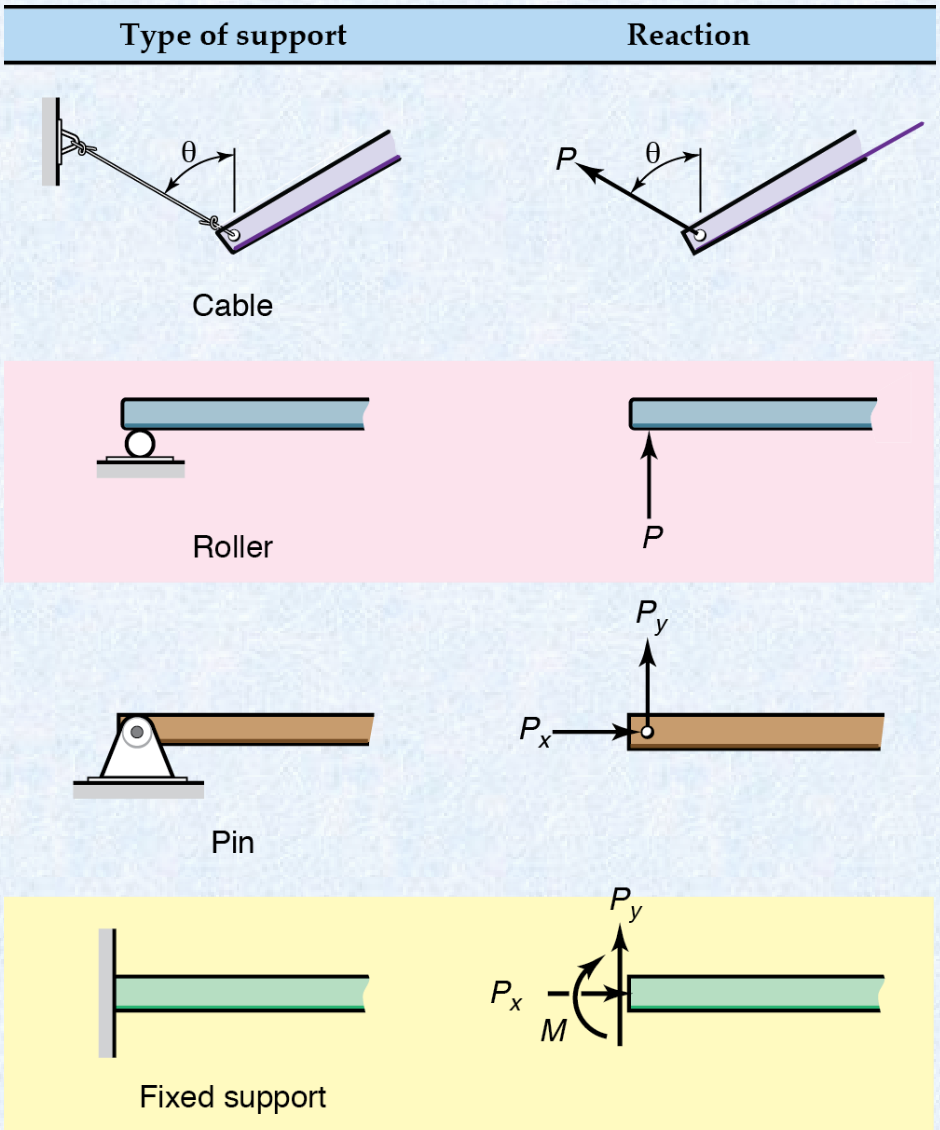


Table 1: Four types of support with their corresponding reactions.

Example 3

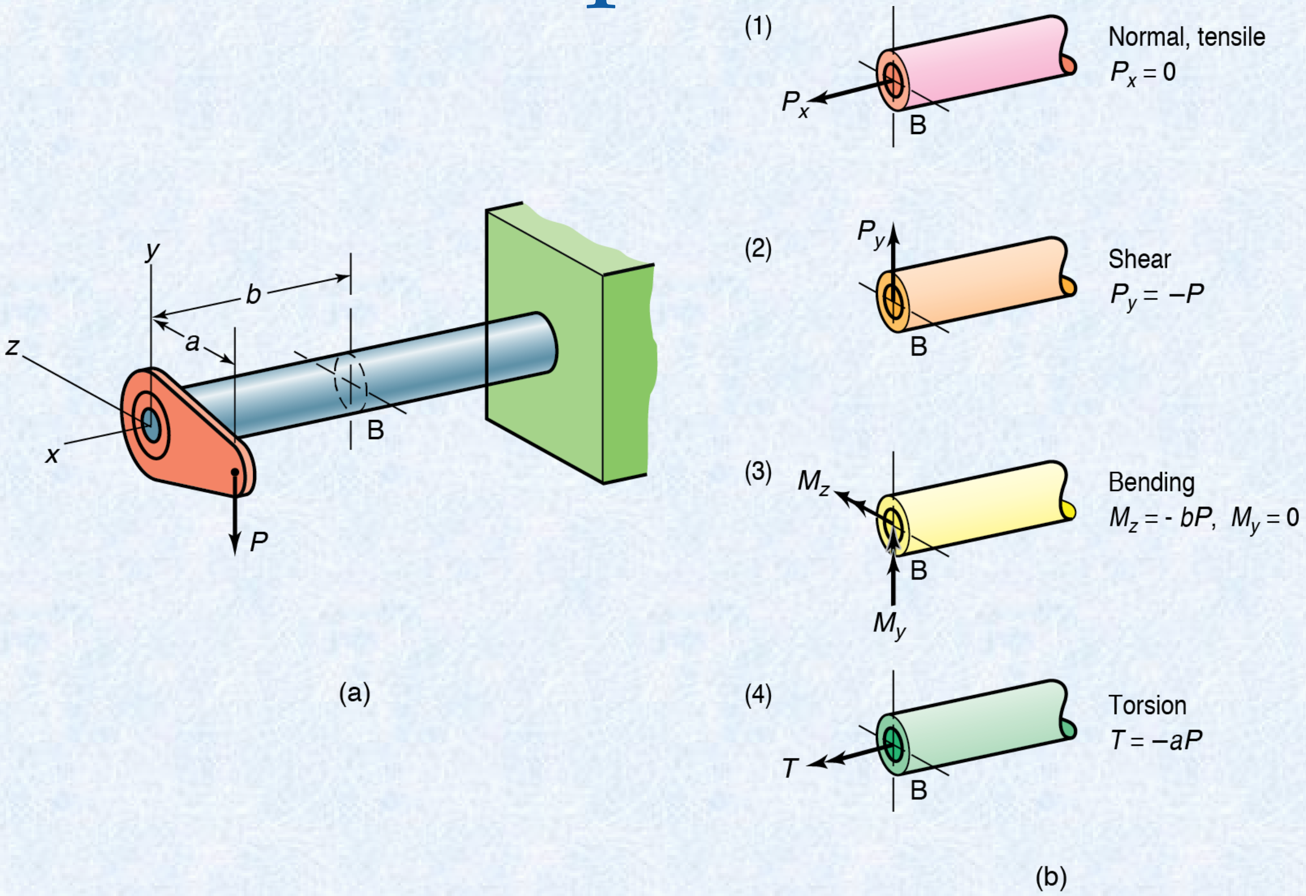


Figure 4: Lever assembly and results. (a) Lever assembly; (b) results showing (1) normal, tensile, (2) shear, (3) bending, (4) torsion on section B of lever assembly.

Example 4

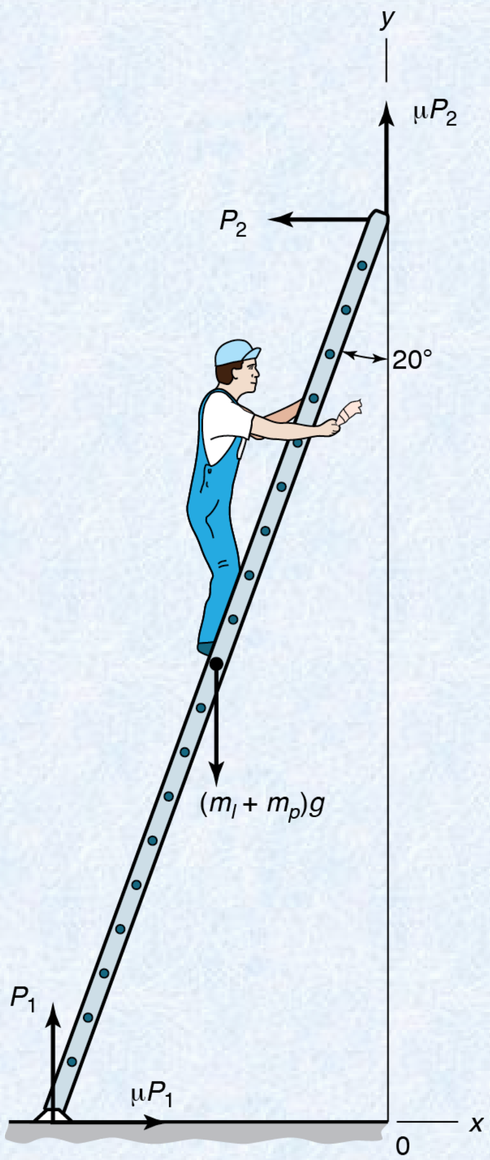


Figure 5: Ladder in contact with a house and the ground while having a painter on the ladder.

Example 5

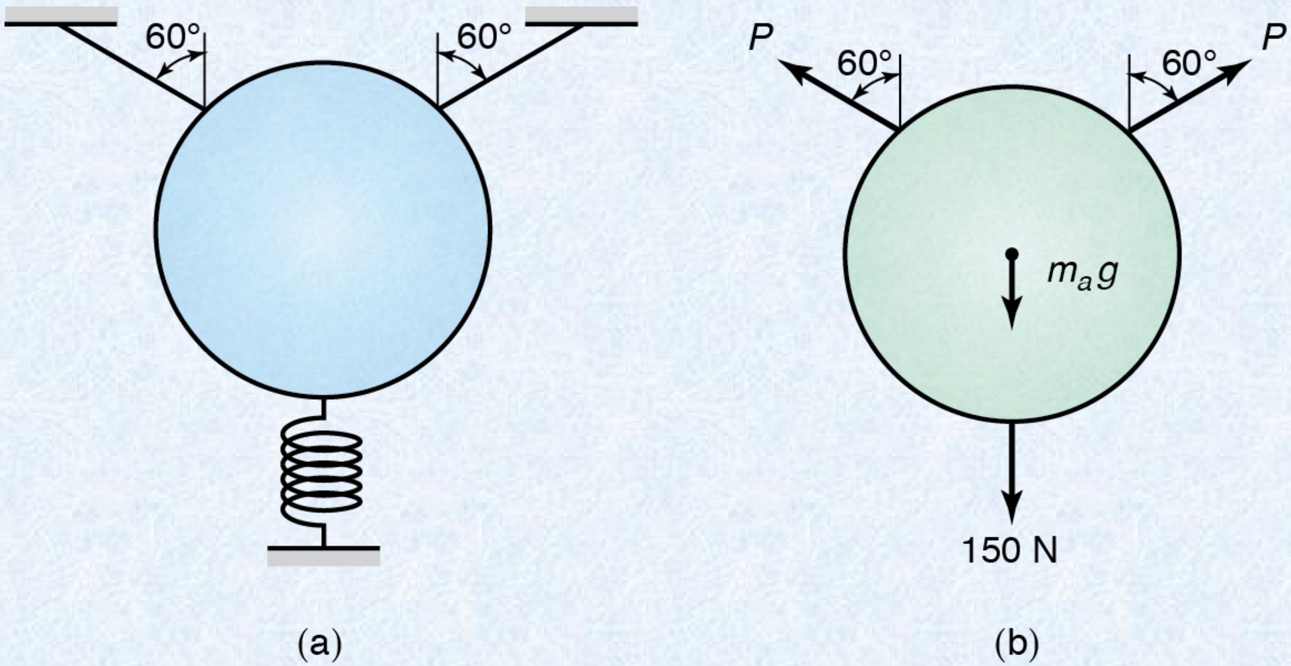


Figure 6: Sphere and applied forces. (a) Sphere supported with wires from top and spring at bottom; (b) free-body diagram of forces acting on sphere.

Example 6

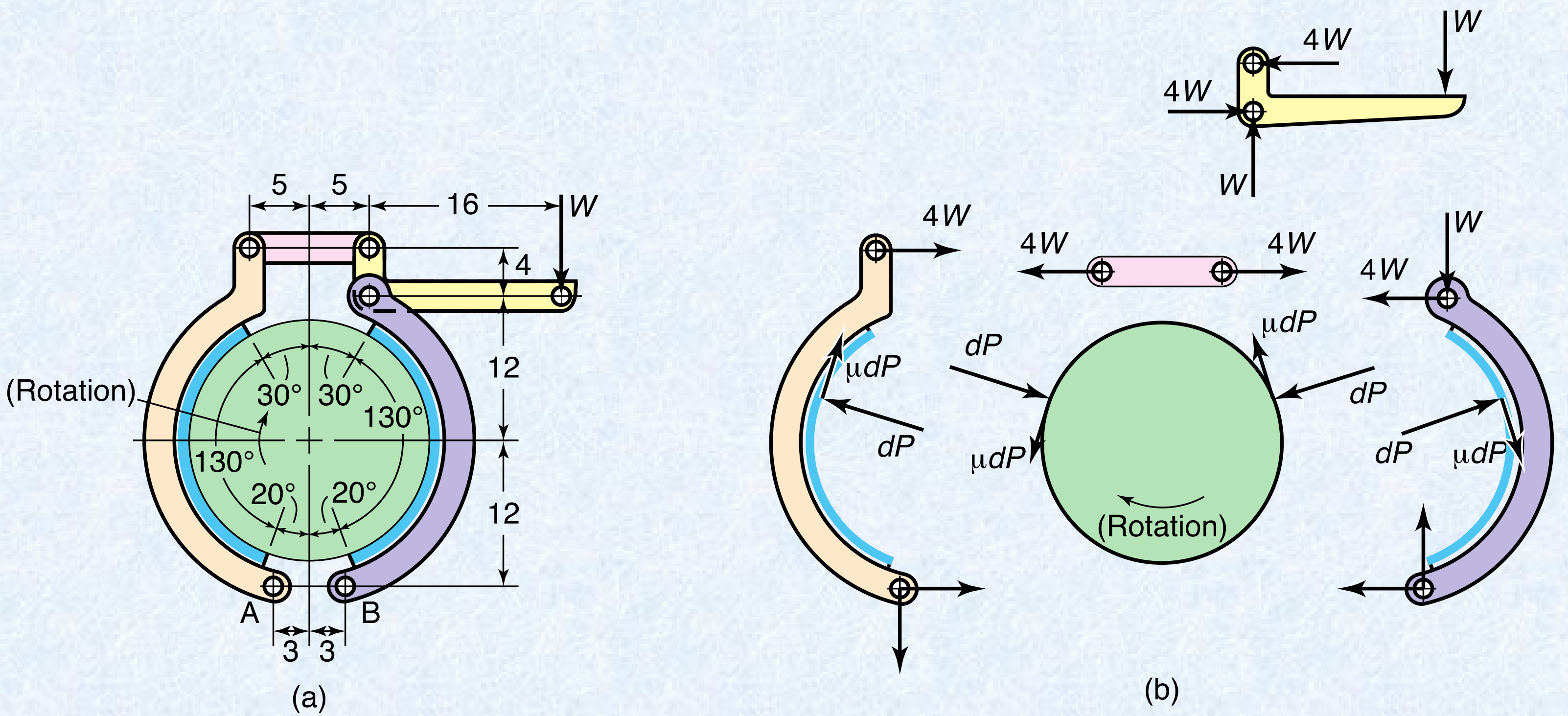


Figure 7: External rim brake and applied forces, considered in Example 2.6. (a) External rim brake; (b) external rim brake with forces acting on each part. (Linear dimensions are in millimeters.)

Beam Supports

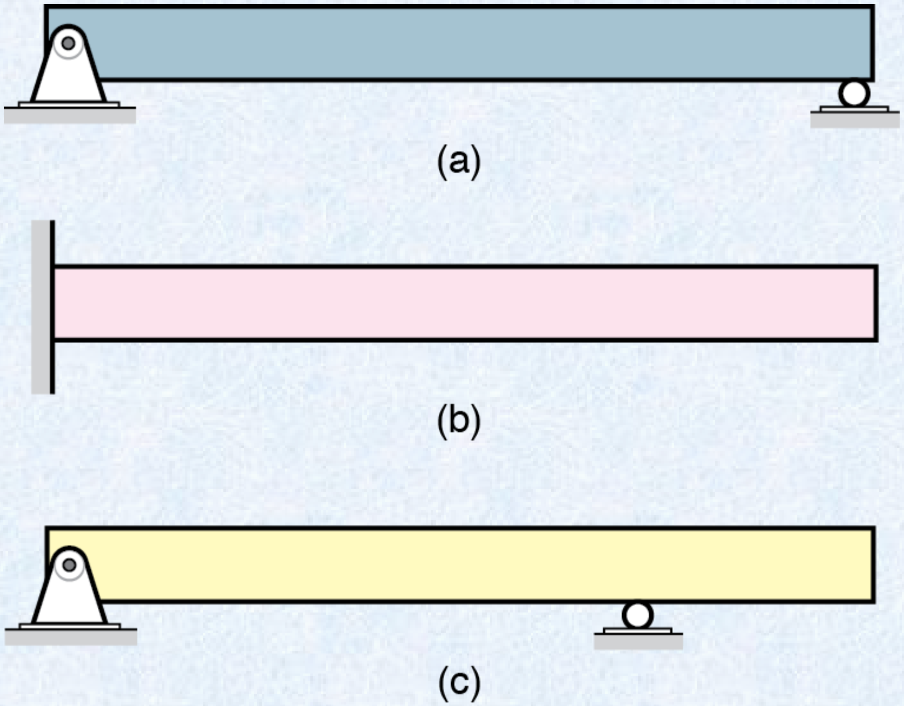


Figure 8: Three types of beam support. (a) Simply supported; (b) cantilevered; (c) overhanging.

Design Procedure 2: Drawing Shear and Moment Diagrams by the Method of Sections

The procedure for drawing shear and moment diagrams by the method of sections is as follows:

1. Draw a free-body diagram and determine all the support reactions. Resolve the forces into components acting perpendicular and parallel to the beam's axis.
2. Choose a position, x, between the origin and the length of the beam, l, thus dividing the beam into two segments. The origin is chosen at the beam's left end to ensure that any x chosen will be positive.
3. Draw a free-body diagram of the two segments and use the equilibrium equations to determine the transverse shear force, V, and the moment, M.
4. Plot the shear and moment functions versus x. Note the location of the maximum moment. Generally, it is convenient to show the shear and moment diagrams directly below the free-body diagram of the beam.
5. Additional sections can be taken as necessary to fully quantify the shear and moment diagrams.

Example 7

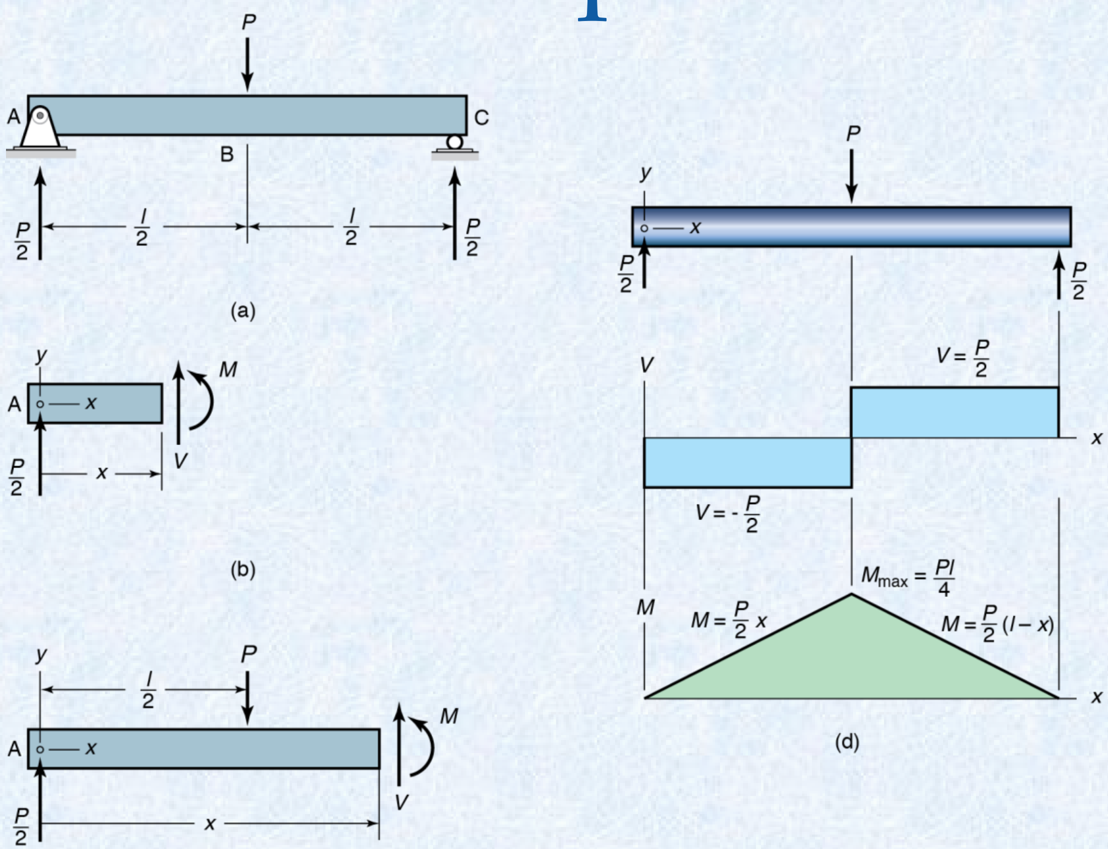


Figure 9: Simply supported bar. (a) Midlength load and reactions; (b) free-body diagram for $0 < x < l/2$; (c) free-body diagram for $l/2 \leq x \leq l$; (d) shear and moment diagrams.

Example 8

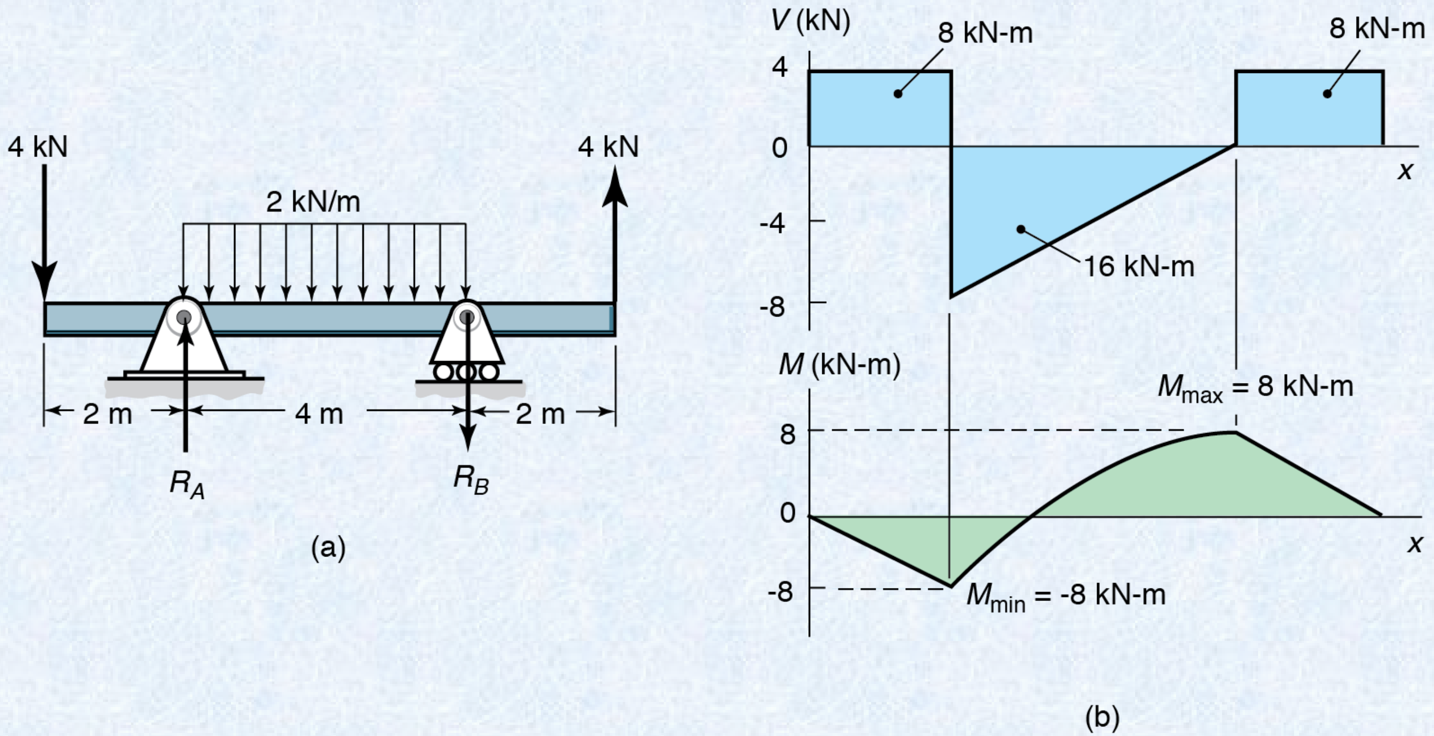


Figure 10: Beam for Example 8. (a) Applied loads and reactions; (b) Shear diagram with areas indicated, and moment diagram with maximum and minimum values indicated.

Design Procedure 3: **Singularity Functions**

Some general rules relating to singularity functions are:

1. If $n > 0$ and the expression inside the angular brackets is positive (i.e., $x \geq a$), then $f_n(x) = (x - a)^n$. Note that the angular brackets to the right of the equal sign in Eq.~(2.6) are now parentheses.
2. If $n > 0$ and the expression inside the angular brackets is negative (i.e., $x < a$), then $f_n(x) = 0$.
3. If $n < 0$, then $f_n(x) = 0$.
4. If $n = 0$, then $f_n(x) = 1$ when $x \geq a$ and $f_n(x) = 0$ when $x < a$.
5. If $n \geq 0$, the integration rule is
$$\int_{-\infty}^{x} \langle x - a \rangle^n = \frac{\langle x - a \rangle^{n+1}}{n+1}.$$

Note that this is the same as if there were parentheses instead of angular brackets.

6. If $n < 0$, the integration rule is
$$\int_{-\infty}^{x} \langle x - a \rangle^n dx = \langle x - a \rangle^{n+1}.$$
7. When $n \geq 1$, then $\frac{d}{dx}\langle x - a \rangle^n = n\langle x - a \rangle^{n-1}.$

Design Procedure 4: Shear and Moment Diagrams by Singularity Functions

The procedure for drawing the shear and moment diagrams by making use of singularity functions is as follows:

1. Draw a free-body diagram with all the applied distributed and concentrated loads acting on the beam, and determine all support reactions. Resolve the forces into components acting perpendicular and parallel to the beam's axis.
2. Write an expression for the load intensity function $q(x)$ that describes all the singularities acting on the beam. Use Table 2.2 as a reference, and make sure to "turn off" singularity functions for distributed loads and the like that do not extend across the full length of the beam.
3. Integrate the negative load intensity function over the beam length to get the shear force. Integrate the negative shear force distribution over the beam length to get the moment, in accordance with Eqs. (2.4) and (2.5).
4. Draw shear and moment diagrams from the expressions developed.

Singularity Functions

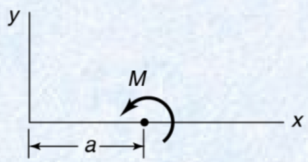
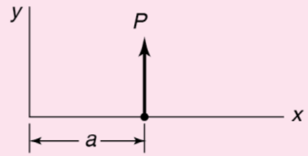
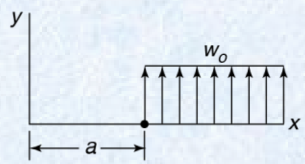
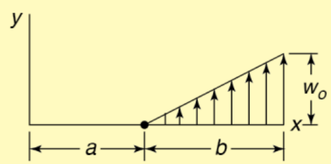
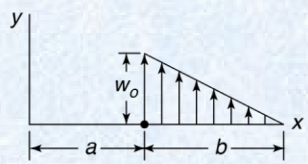
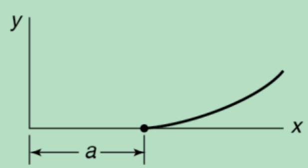
Singularity	Graph of $q(x)$	Expression for $q(x)$
Concentrated moment		$q(x) = M\langle x-a\rangle^{-2}$
Concentrated force		$q(x) = P\langle x-a\rangle^{-1}$
Unit step		$q(x) = w_o\langle x-a\rangle^{0}$
Ramp		$q(x) = \frac{w_o}{b}\langle x-a\rangle^{1}$
Inverse ramp		$q(x) = w_o\langle x-a\rangle^{0} - \frac{w_o}{b}\langle x-a\rangle^{1}$
Parabolic shape		$q(x) = \langle x-a\rangle^{2}$

Table 2: Singularity and load intensity functions with corresponding graphs and expressions.

Example 9

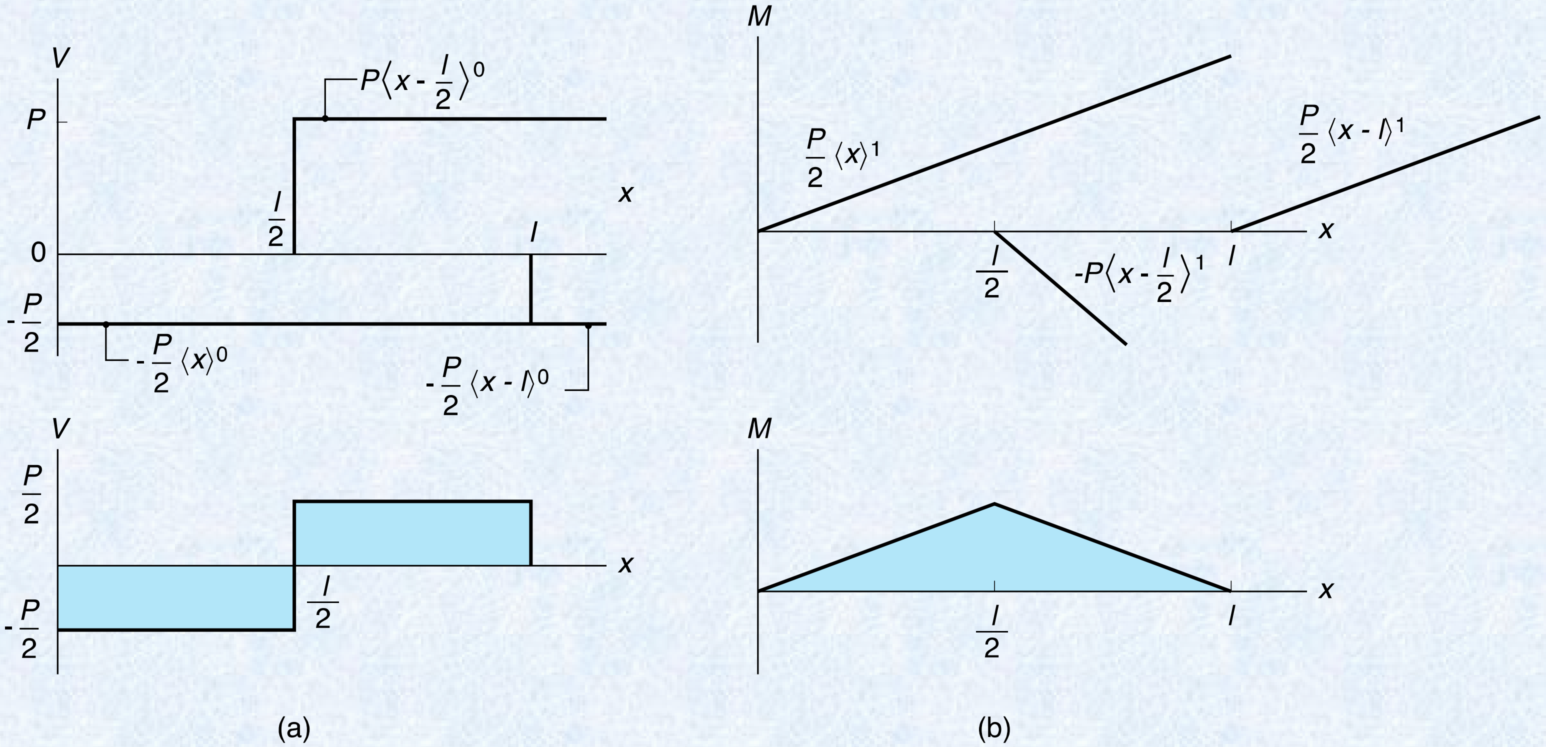


Figure 11: Beam for Example 8. (a) Applied loads and reactions; (b) Shear diagram with areas indicated, and moment diagram with maximum and minimum values indicated.

Example 10

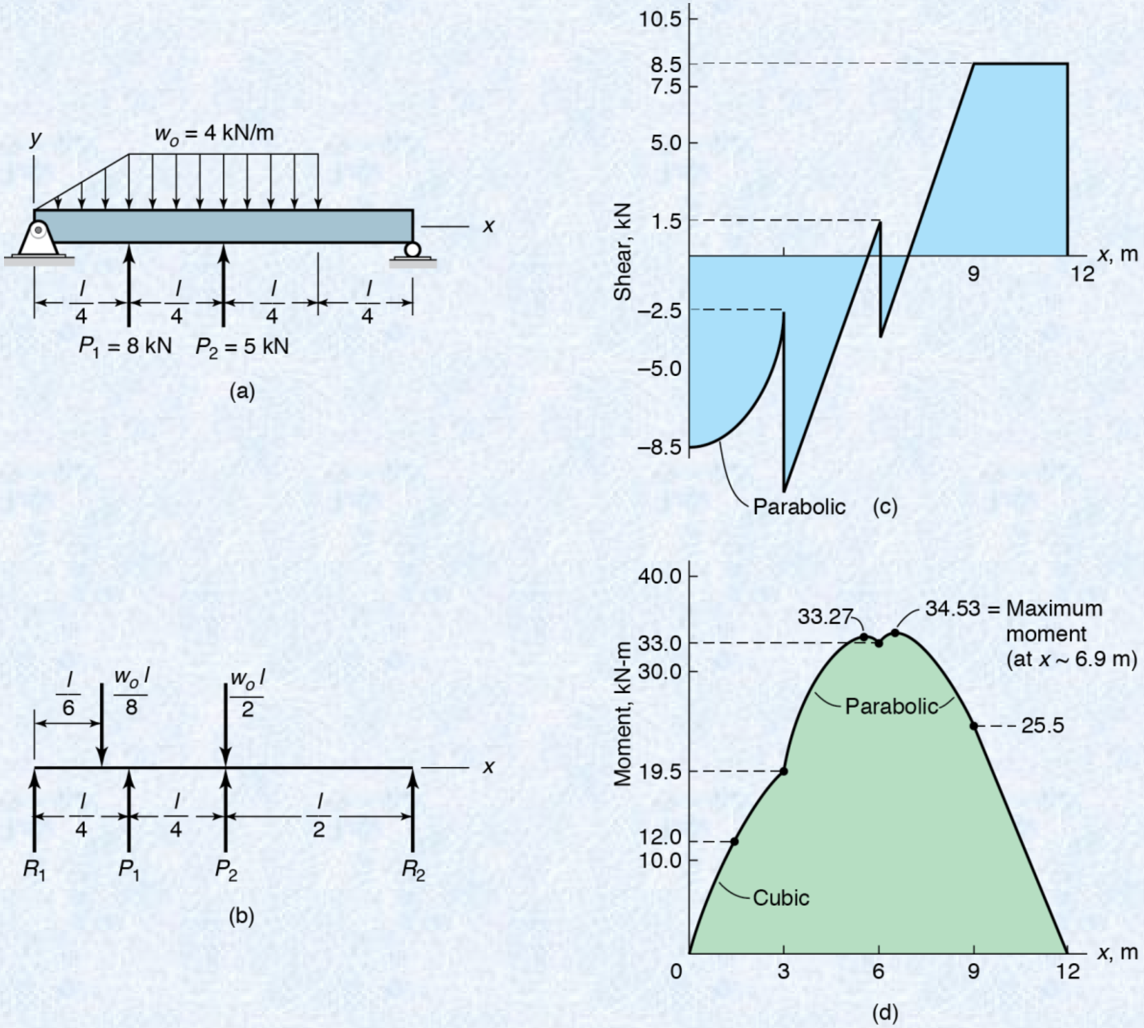


Figure 12: Simply supported beam examined in Example 10. (a) Forces acting on beam when $P_1 = 8$ kN, $P_2 = 5$ kN; $w_o = 4$ kN/m; $l = 12$ m; (b) free-body diagram showing resulting forces; (c) shear and (d) moment diagrams.

Example 11

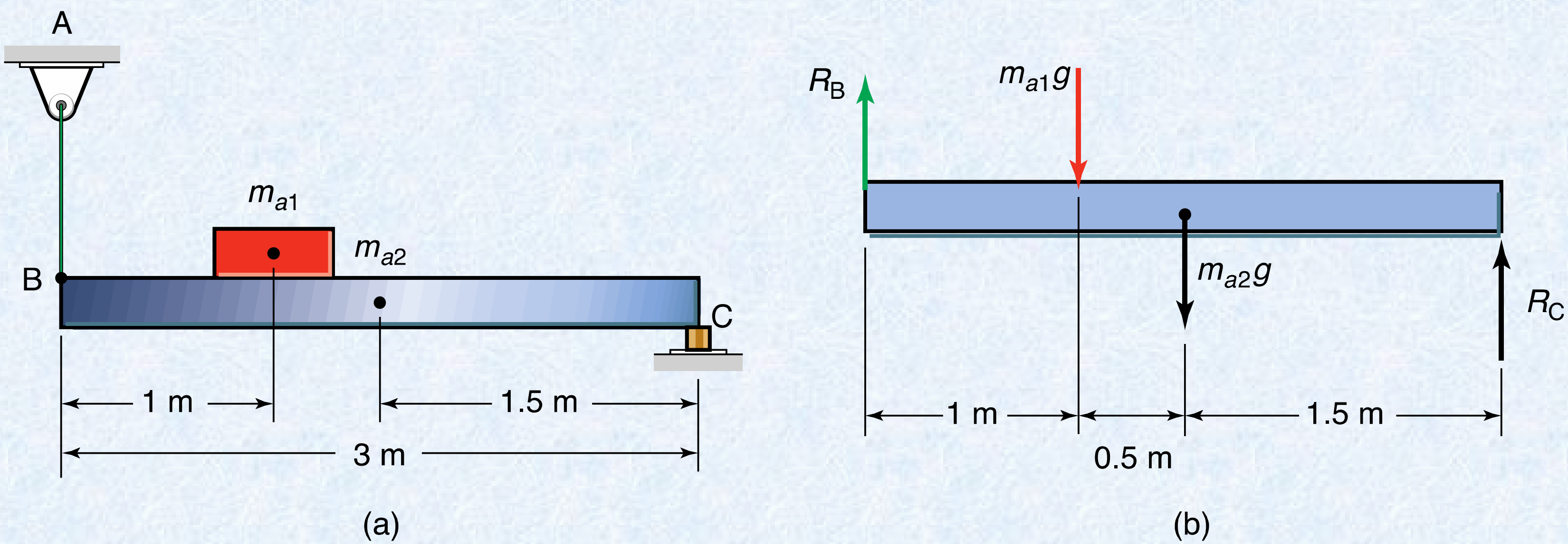


Figure 13: Figures used in Example 11. (a) Load assembly drawing; (b) free-body diagram.

3D Stress Element

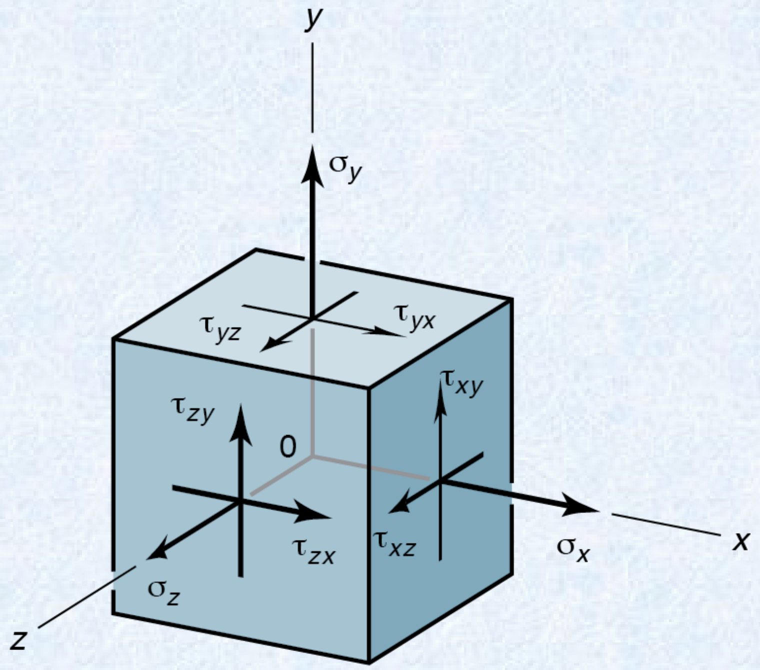


Figure 14: Stress element showing general state of three-dimensional stress with origin placed in center of element.

Normal stress:

$$\sigma_{\text{avg}} = \frac{\text{Average force}}{\text{Cross-sectional area}} = \frac{P}{A}.$$

Stress tensor:

$$\mathbf{S} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix},$$

2D Stress Element

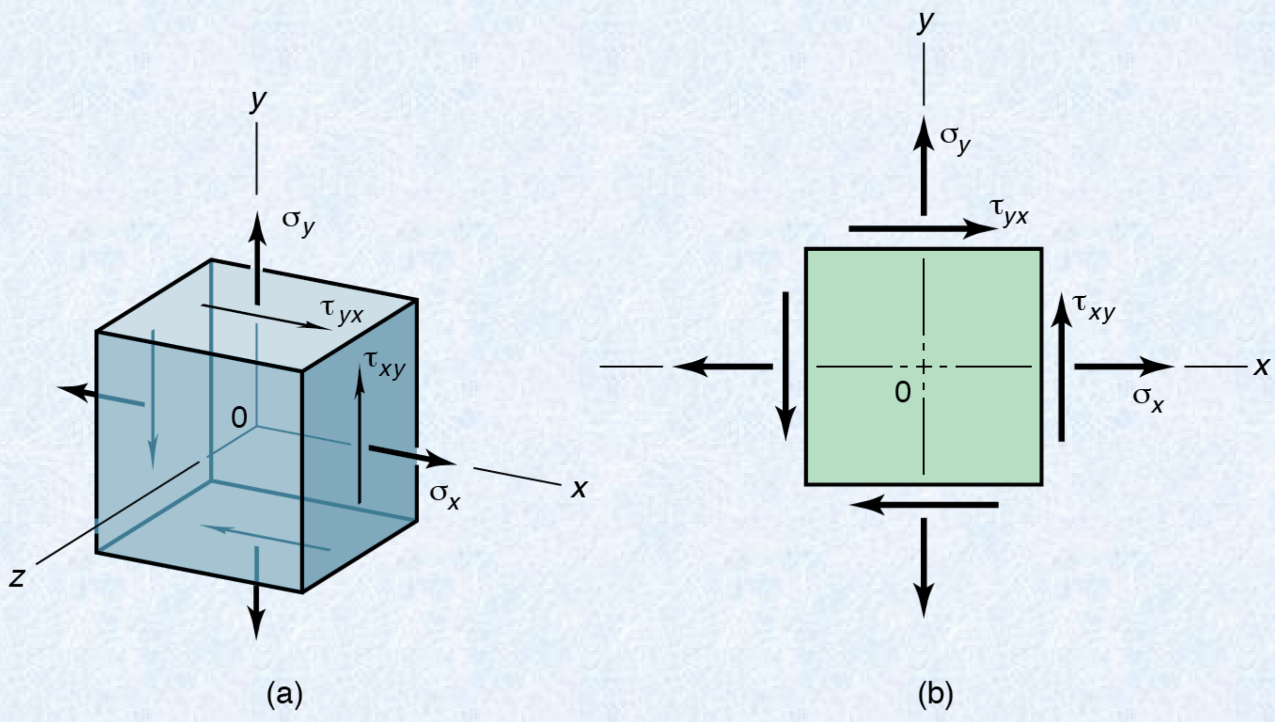


Figure 15: Stress element showing two-dimensional state of stress. (a) Three-dimensional view; (b) plane view.

Equivalent Stress States

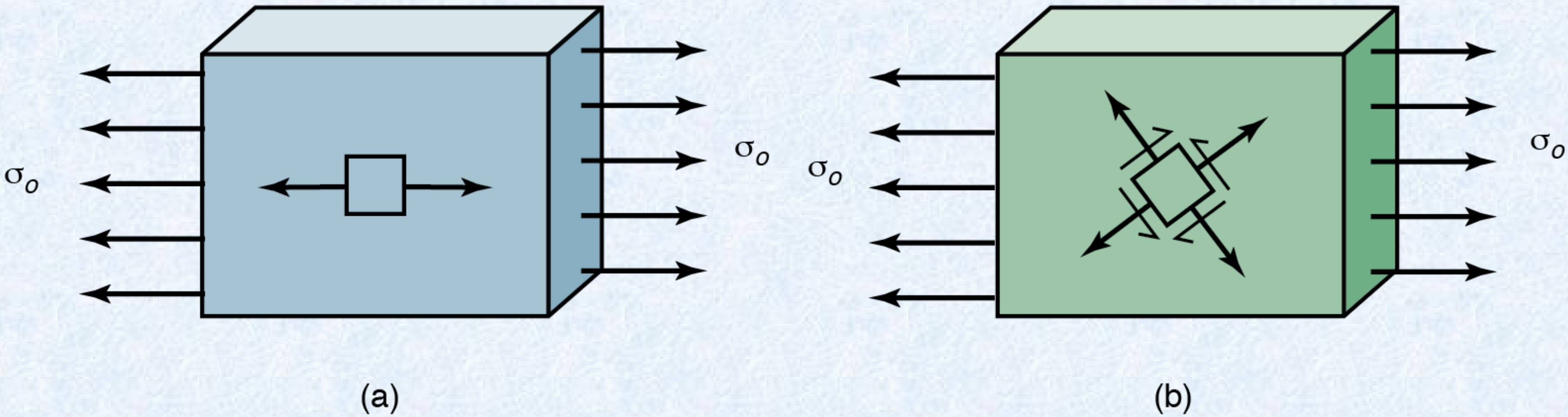
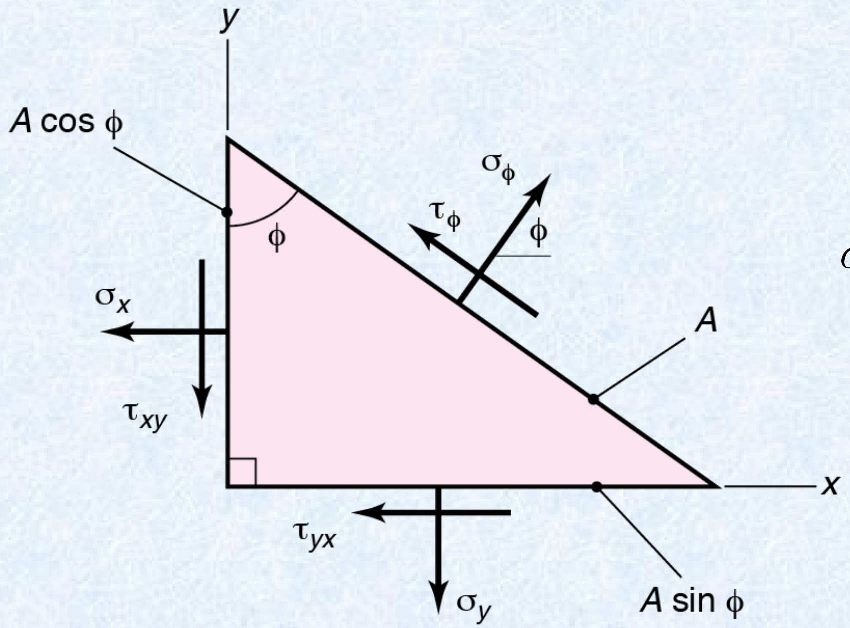


Figure 16: Illustration of equivalent stress states; (a) Stress element oriented in the direction of applied stress; (b) stress element oriented in different (arbitrary) direction.

Stress on an Oblique Plane



Stress transformation equations:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\tau_{xy}^2 + \frac{(\sigma_x - \sigma_y)^2}{4}}$$

Figure 17: Stresses in an oblique plane at an angle φ.

Design Procedure 5: Mohr’s Circle

The steps in constructing and using Mohr's circle in two dimensions are as follows:

1. Calculate the plane stress state for any *x*-*y* coordinate system so that σ_x, σ_y, and τ_{xy} are known.
2. The center of the Mohr's circle can be placed at
$$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$$
3. Two points diametrically opposite to each other on the circle correspond to the points (σ_x, -τ_{xy}) and (σ_y, τ_{xy}). Using the center and either point allows one to draw the circle.
4. The radius of the circle can be calculated from stress transformation equations or through geometry by using the center and one point on the circle. For example, the radius is the distance between points (σ_x, -τ_{xy}) and the center, which directly leads to
$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Design Procedure 5: Mohr's Circle (cont.)

5. The principal stresses have the values $\sigma_{1,2}$ = center ± radius.
6. The maximum shear stress equals the radius.
7. The principal axes can be found by calculating the angle between the *x*-axis in the Mohr's circle plane and the point (σ_x, -τ_{xy}). The principal axes in the real plane are rotated one-half this angle in the same direction relative to the *x*-axis in the real plane.
8. The stresses in an orientation rotated φ from the *x*-axis in the real plane can be read by traversing an arc of 2φ in the same direction on the Mohr's circle from the reference points (σ_x, - τ_{xy}) and (σ_y, τ_{xy}). The new points on the circle correspond to the new stresses ($\sigma_{x'}$, - τ_{xy}) and ($\sigma_{y'}$, τ_{xy}), respectively.

Mohr’s Circle

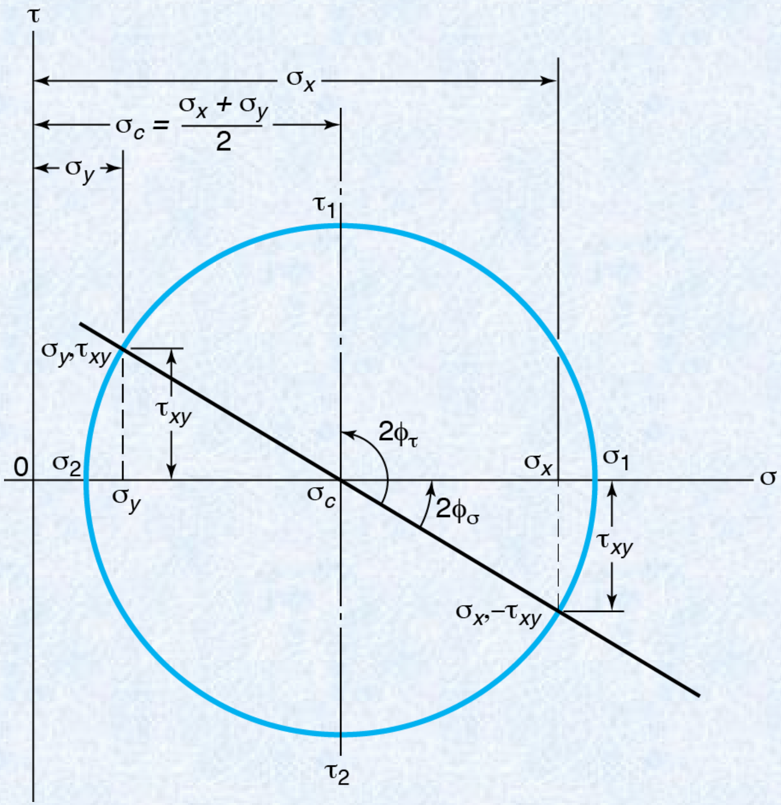


Figure 18: Mohr's circle diagram of Eqs.

Center at:

$$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$$

Radius:

$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Example 14

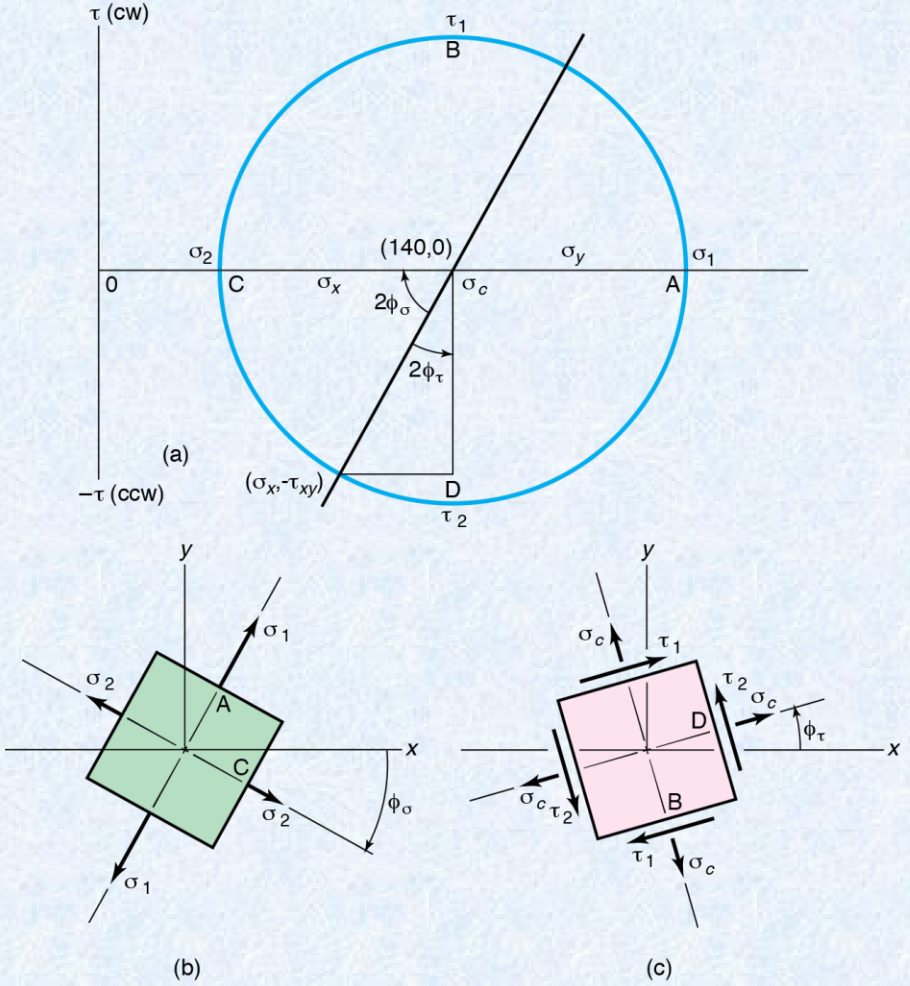


Figure 19: Results from Example 14. (a) Mohr's circle diagram; (b) stress element for principal normal stress shown in x-y coordinates; (c) stress element for principal shear stresses shown in x-y coordinates.

Mohr’s Circle for Triaxial Stresses

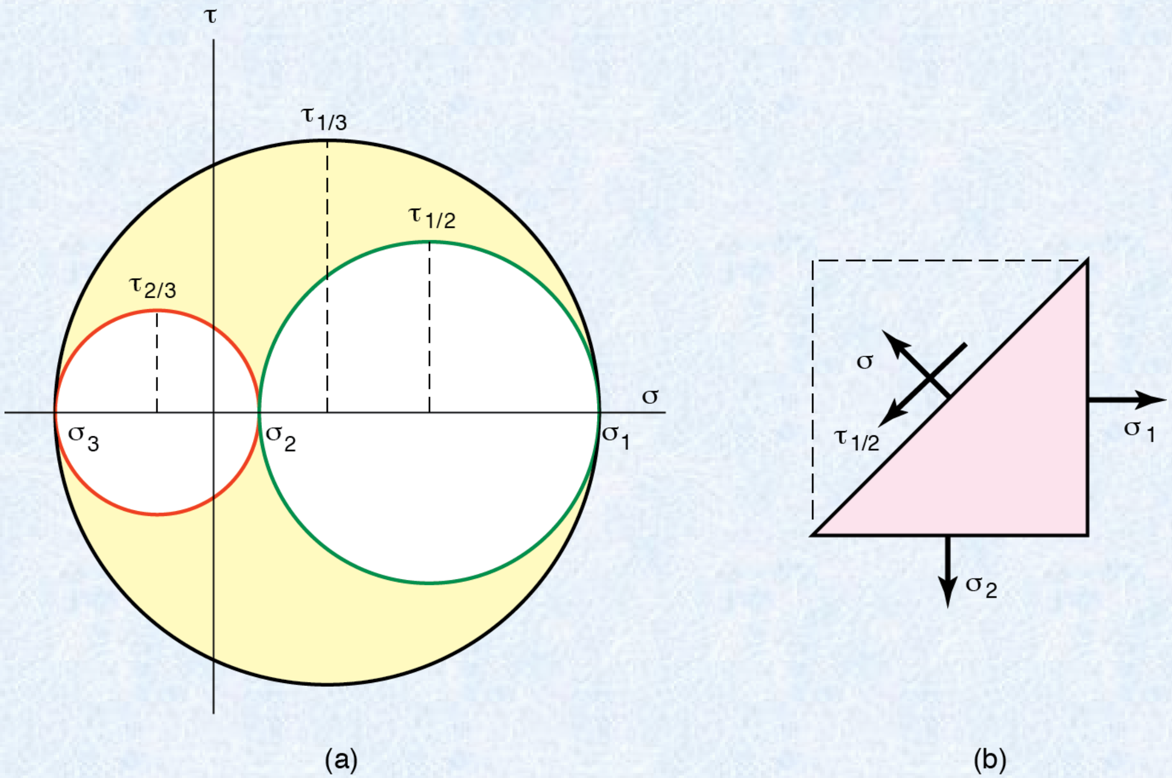


Figure 20: Mohr's circle for triaxial stress state. (a) Mohr's circle representation; (b) principal stresses on two planes.

Example 15

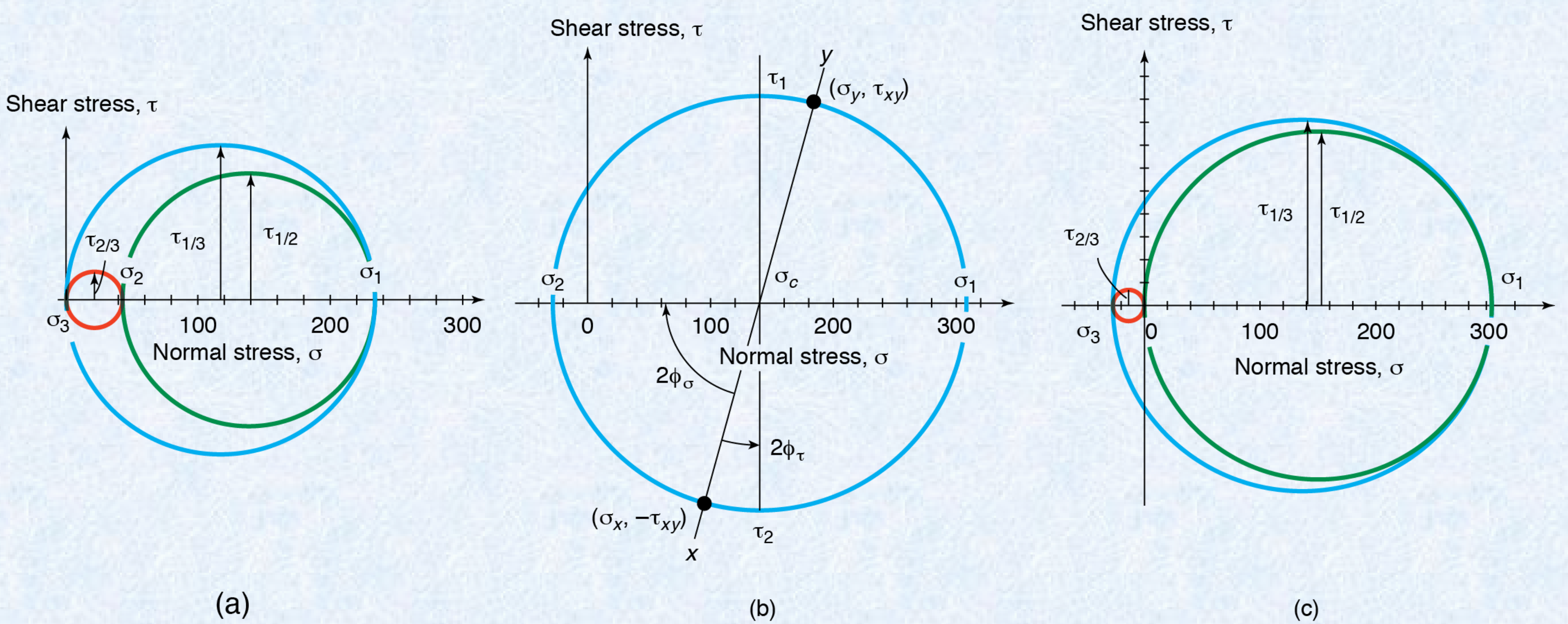


Figure 21: Mohr's circle diagrams for Example 15. (a) Triaxial stress state when $\sigma_1 =$ 234.3 MPa, $\sigma_2 = 457$ MPa and $\sigma_3 = 0$; (b) biaxial stress state when $\sigma_1 = 307.6$ MPa and $\sigma_2 =$ -27.6 MPa; (c) triaxial stress state when $\sigma_1 = 307.6$ MPa, $\sigma_2 = 0$, and $\sigma_3 = -27.6$ MPa.

Octahedral Stresses

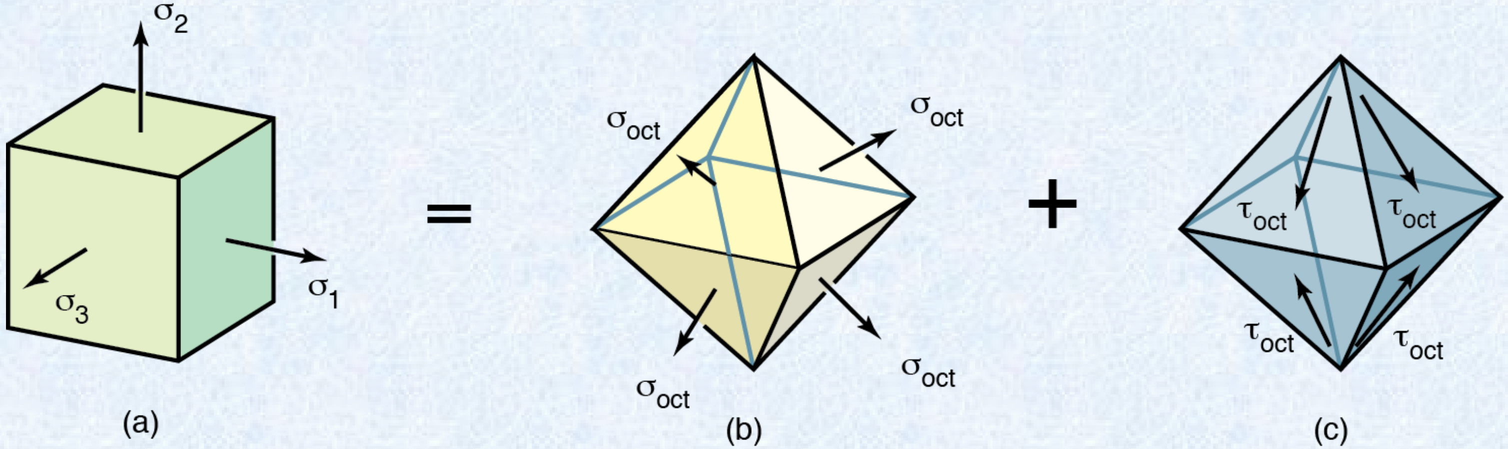


Figure 22: Stresses acting on octahedral planes. (a) General state of stress. (b) normal stress; (c) octahedral shear stress.

Octahedral stresses:

$$\sigma_{\rm oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\tau_{\rm oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \frac{2}{3}\left[\tau_{1/2}^2 + \tau_{2/3}^2 + \tau_{1/3}^2\right]^{1/2}.$$

Normal Strain

Normal strain: $$\epsilon_{\text{avg}} = \frac{\delta}{l} = \frac{\text{Average elongation}}{\text{Original length}}$$

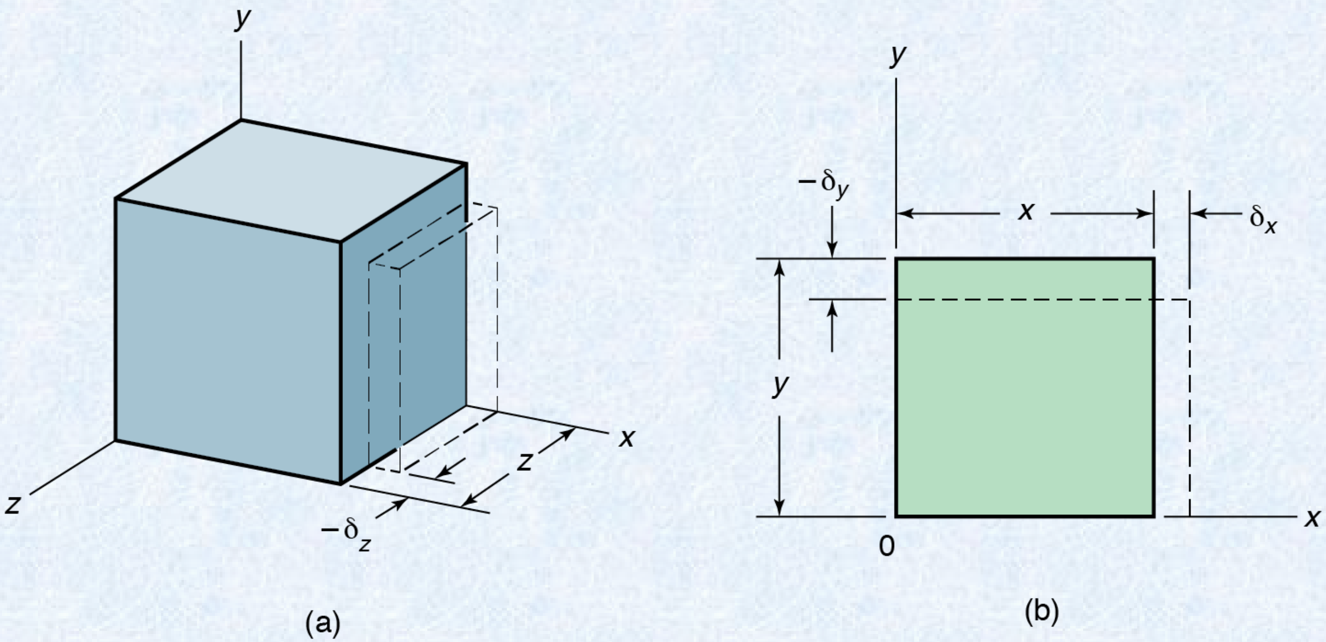


Figure 23: Normal strain of cubic element subjected to uniform tension in x-direction. (a) Three-dimensional view; (b) two-dimensional (or plane) view.

Shear Strain

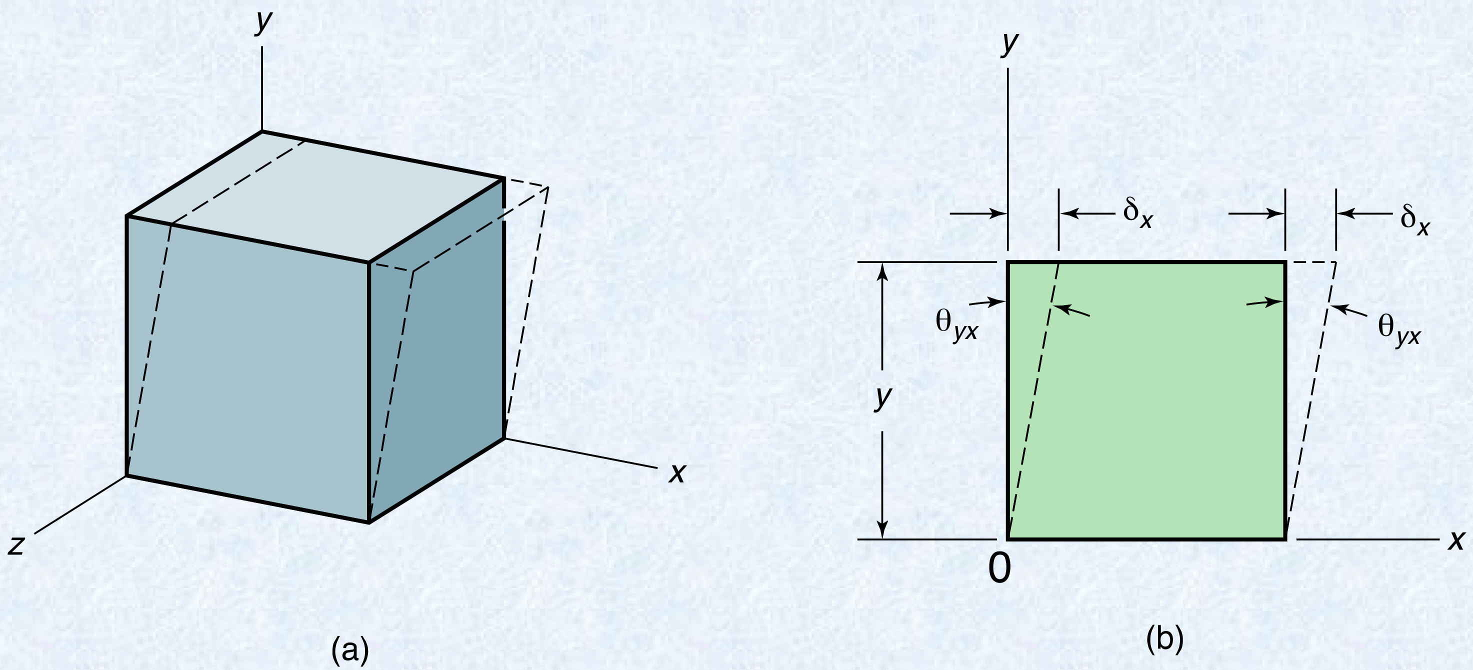


Figure 24: Shear strain of cubic element subjected to shear stress. (a) Three-dimensional view; (b) two-dimensional (or plane) view.

Plane Strain Element

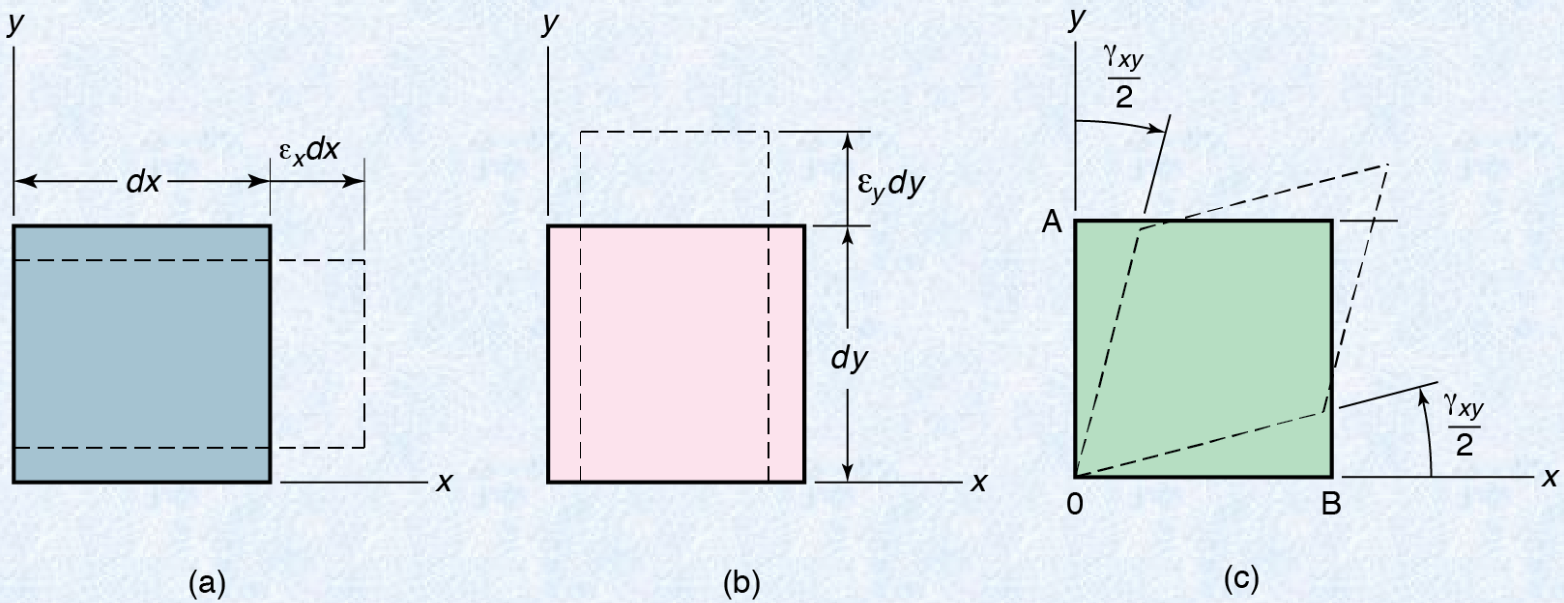


Figure 25: Graphical depiction of plane strain element. (a) Normal strain ε_x; (b) normal strain ε_y; and (c) shear strain γ_{xy}.

Example 18

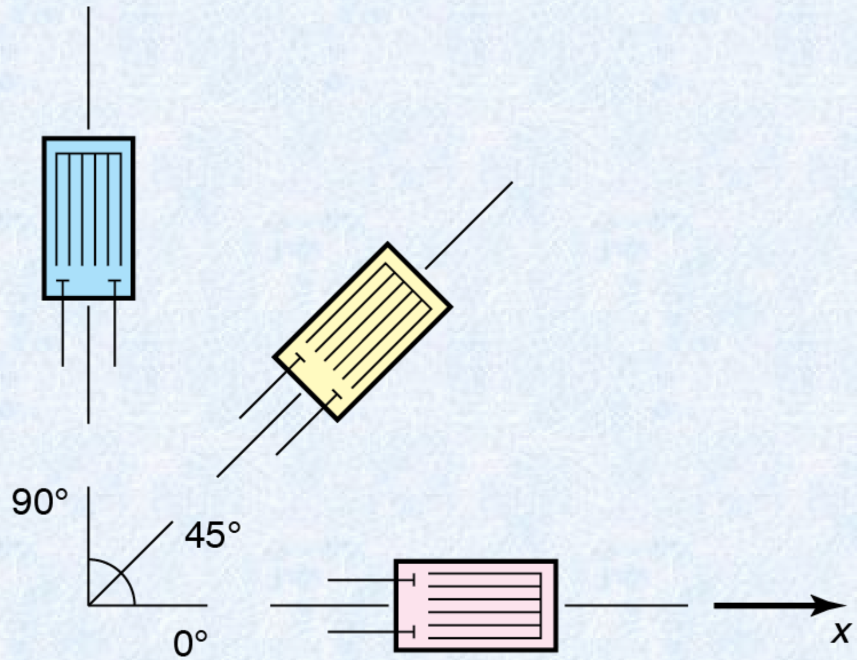


Figure 26: Strain gage rosette used in Example 18.