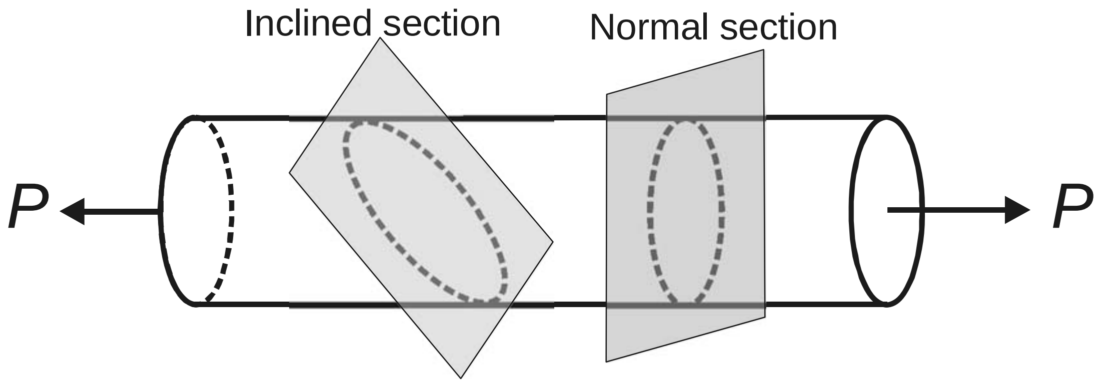
Plane Stress
Transformation Equations

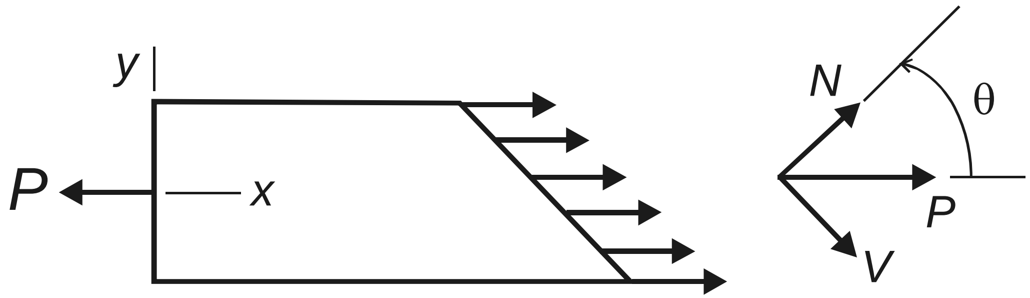
Stress elements and plane stress.
Stresses on inclined sections.
Transformation equations.
Principal stresses, angles, and planes.
Maximum shear stress.

Normal and shear stresses on inclined sections

To obtain a complete picture of the stresses in a bar, we must consider the stresses acting on an “inclined” (as opposed to a “normal”) section through the bar.

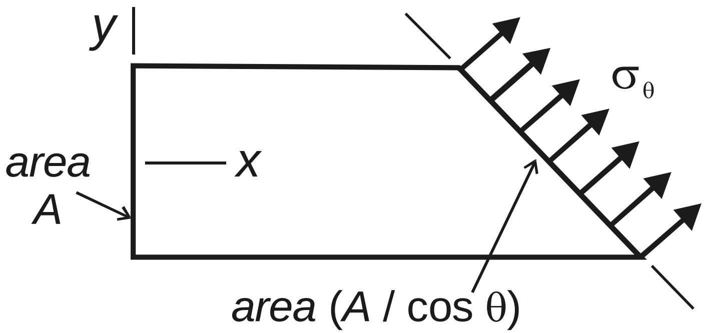


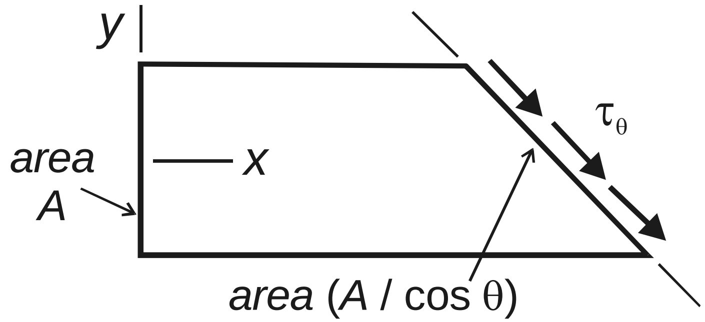
Because the stresses are the same throughout the entire bar, the stresses on the sections are uniformly distributed.

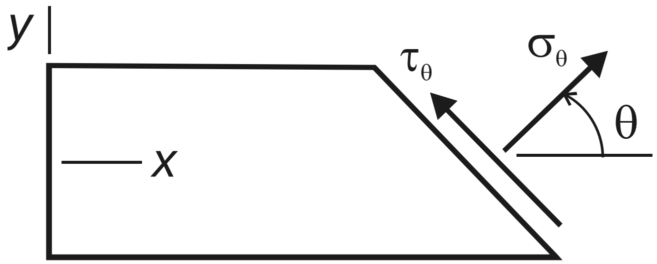


The force P can be resolved into components:
Normal force N perpendicular to the inclined plane, $N = P \cos \theta$
Shear force V tangential to the inclined plane $V = P \sin \theta$

If we know the areas on which the forces act, we can calculate the associated stresses.







$$\sigma_\theta = \frac{Force}{Area} = \frac{N}{Area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta = \frac{\sigma_x}{2}\left(1+\cos 2\theta\right)$$

$\sigma_{max} = \sigma_x$ occurs when $\theta = 0°$

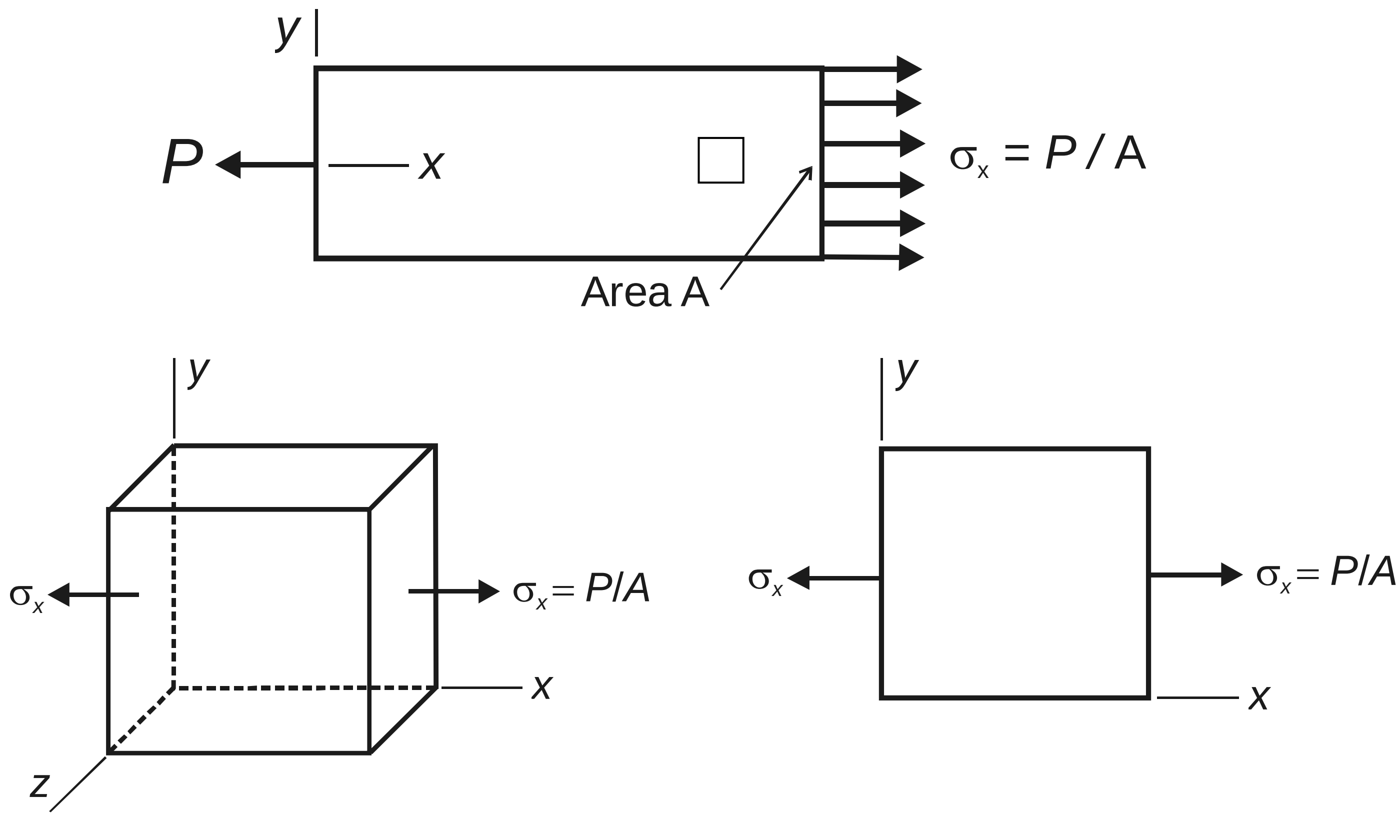
$$\tau_\theta = \frac{Force}{Area} = \frac{-V}{Area} = \frac{-P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta$$

$$\tau_\theta = -\sigma_x \sin\theta\cos\theta = -\frac{\sigma_x}{2}\left(\sin 2\theta\right)$$

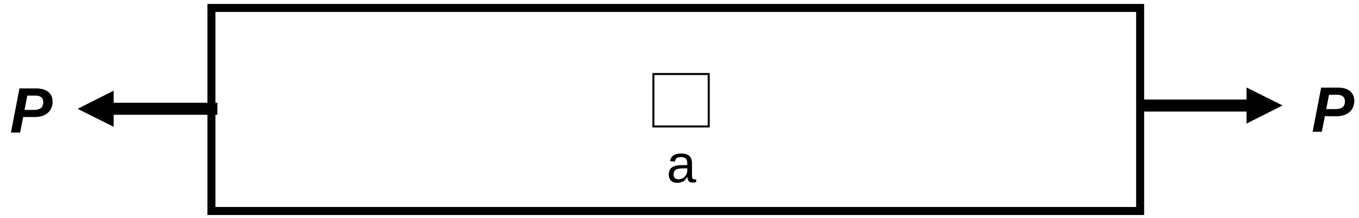
$\tau_{max} = \pm\, \sigma_x/2$ occurs when $\theta = -/+\ 45°$

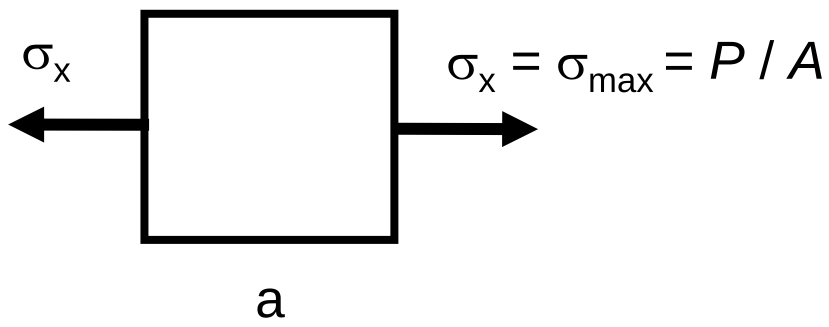
Introduction to stress elements

Stress elements are a useful way to represent stresses acting at some point on a body. Isolate a small element and show stresses acting on all faces. Dimensions are “infinitesimal”, but are drawn to a large scale.



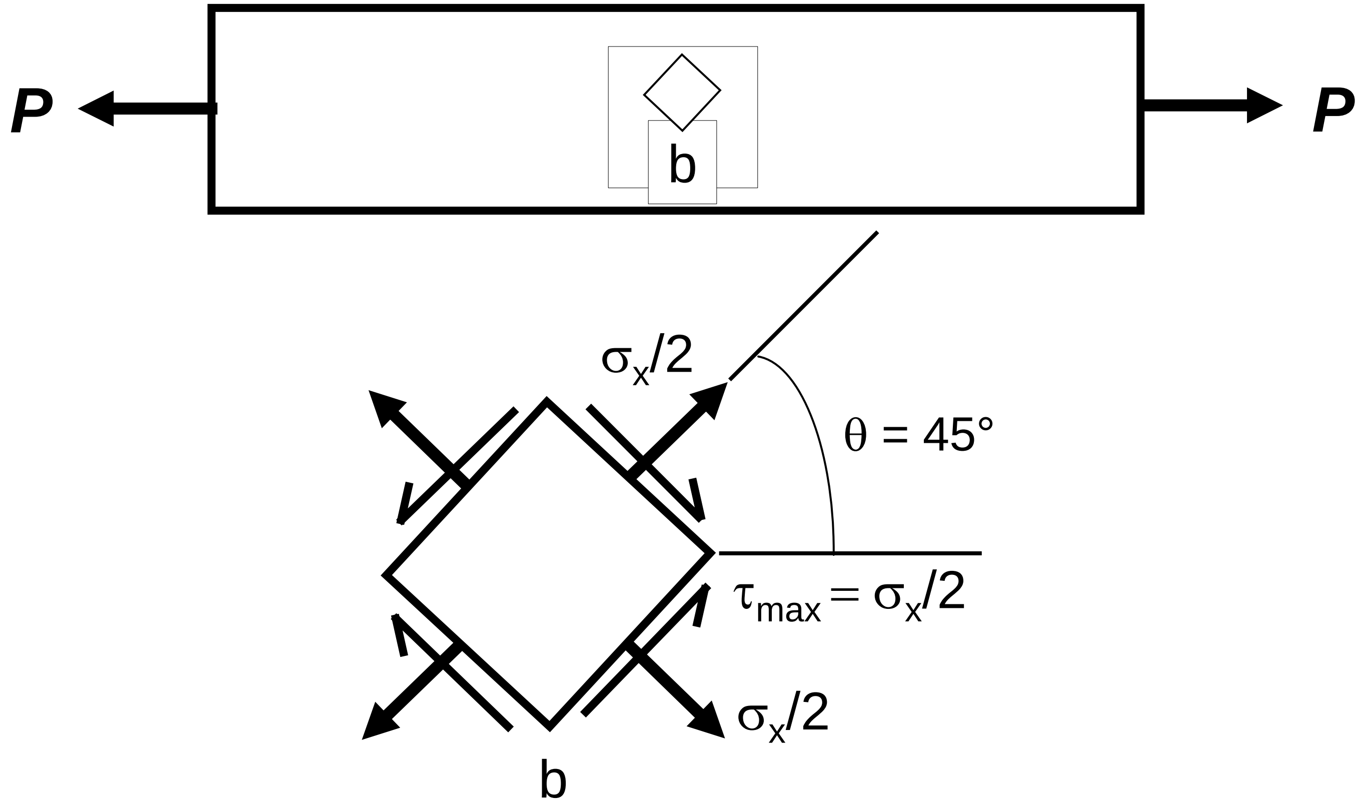
Maximum stresses on a bar in tension





Maximum normal stress,
Zero shear stress

Maximum stresses on a bar in tension

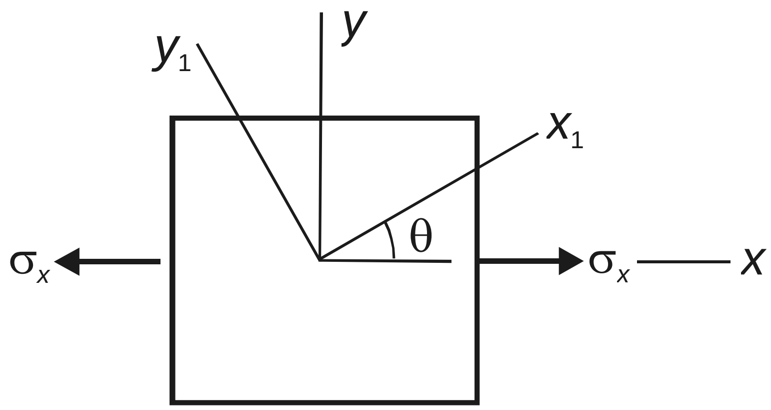


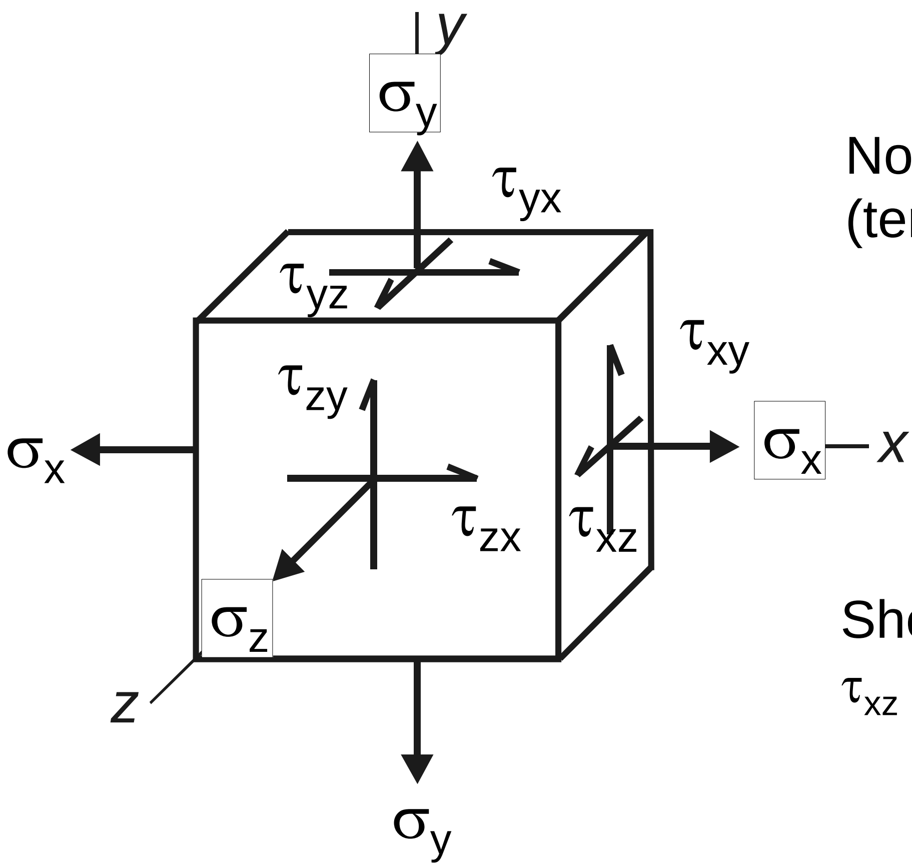
Maximum shear stress,
Non-zero normal stress

Stress Elements and Plane Stress

When working with stress elements, keep in mind that only one intrinsic state of stress exists at a point in a stressed body, regardless of the orientation of the element used to portray the state of stress.

We are really just rotating axes to represent stresses in a new coordinate system.





Normal stresses σ_x, σ_y, σ_z
(tension is positive)

Shear stresses $\tau_{xy} = \tau_{yx}$,
$\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$

Sign convention for τ_{ab}
Subscript a indicates the “face” on which the stress acts
(positive *x* “face” is perpendicular to the positive *x* direction)
Subscript b indicates the direction in which the stress acts
Strictly $\sigma_x = \sigma_{xx}$, $\sigma_y = \sigma_{yy}$, $\sigma_z = \sigma_{zz}$

When an element is in **plane stress** in the *xy* plane, only the *x* and *y* faces are subjected to stresses ($\sigma_z = 0$ and $\tau_{zx} = \tau_{xz} = \tau_{zy} = \tau_{yz} = 0$).

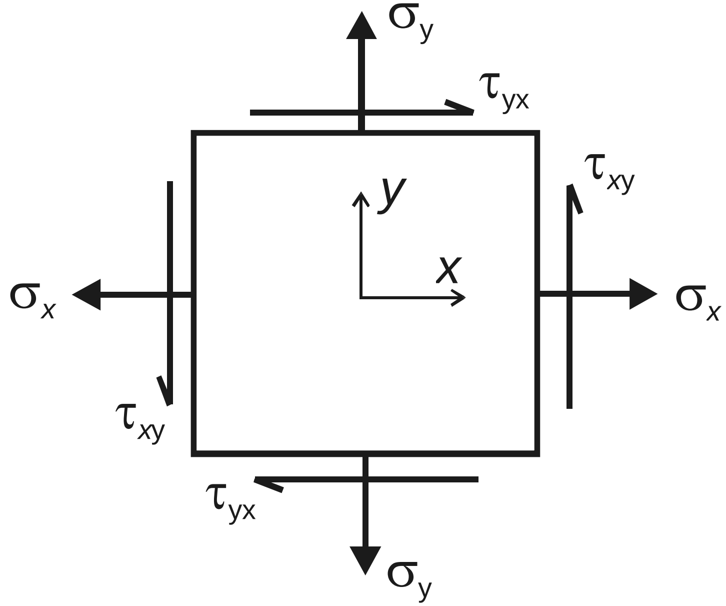
Such an element could be located on the free surface of a body (no stresses acting on the free surface).

Plane stress element in 2D

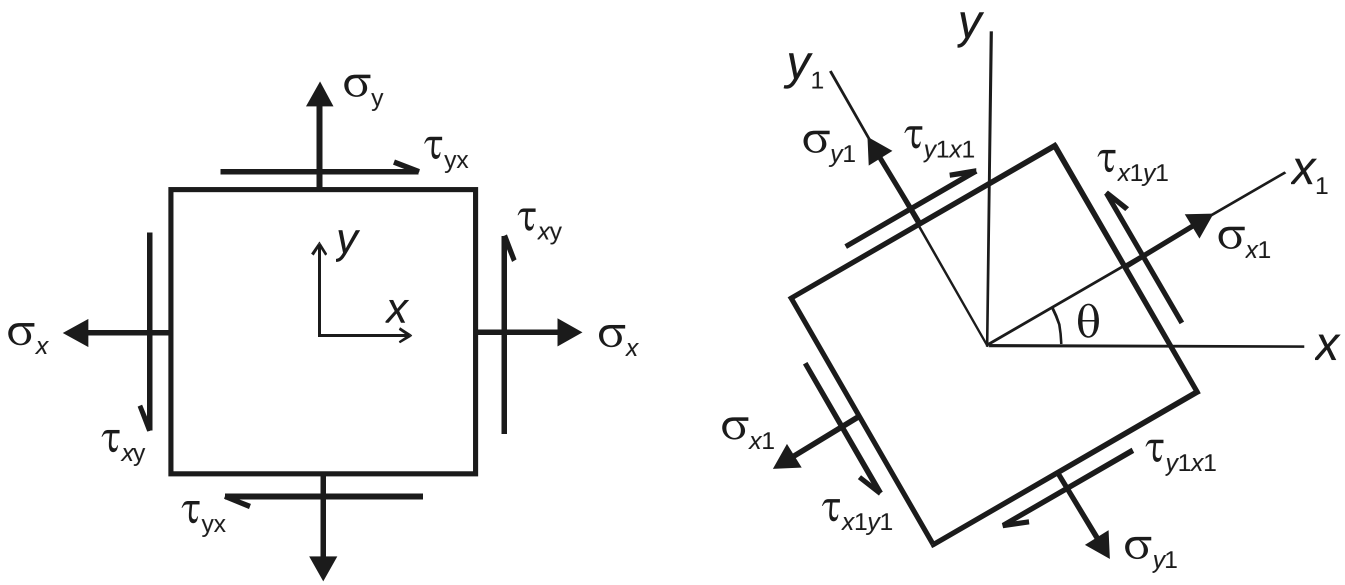
σ_x, σ_y

$\tau_{xy} = \tau_{yx}$

$\sigma_z = 0$



Stresses on Inclined Sections

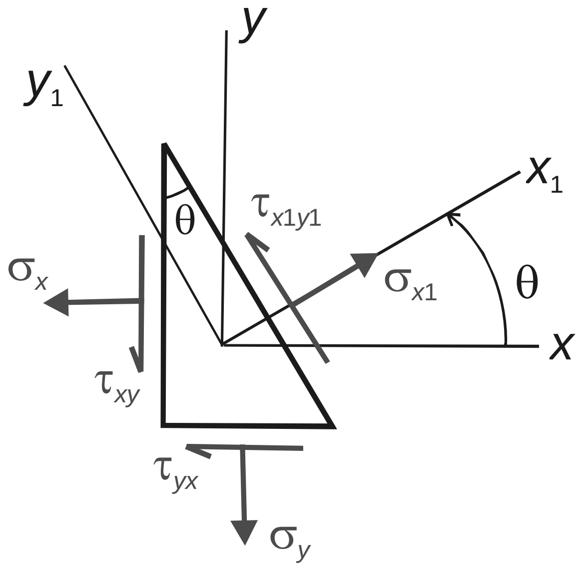


The stress system is known in terms of coordinate system *xy*. We want to find the stresses in terms of the rotated coordinate system x_1y_1.

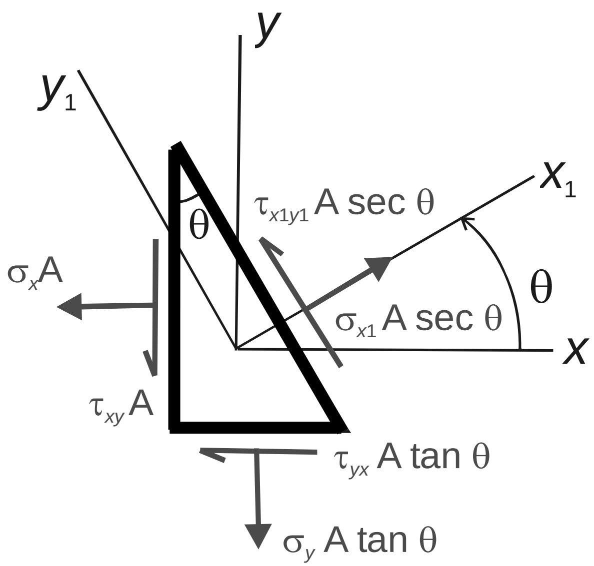
Why? A material may yield or fail at the maximum value of σ or τ. This value may occur at some angle other than $\theta = 0$. (Remember that for uni-axial tension the maximum shear stress occurred when $\theta = 45$ degrees.)

Transformation Equations

Stresses



Forces

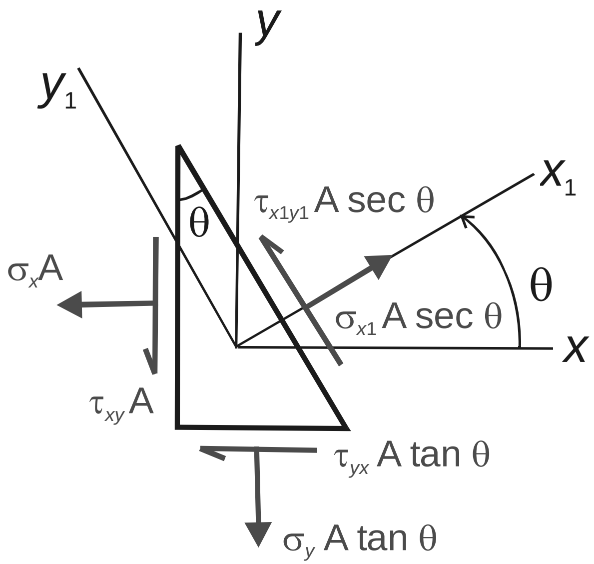


Forces can be found from stresses if the area on which the stresses act is known. Force components can then be summed.

Left face has area A.

Bottom face has area A tan θ.

Inclined face has area A sec θ.



Sum forces in the x_1 direction :

$$\sigma_{x1} A \sec\theta - (\sigma_x A)\cos\theta - (\tau_{xy} A)\sin\theta - (\sigma_y A\tan\theta)\sin\theta - (\tau_{yx} A\tan\theta)\cos\theta = 0$$

Sum forces in the y_1 direction :

$$\tau_{x1y1} A \sec\theta + (\sigma_x A)\sin\theta - (\tau_{xy} A)\cos\theta - (\sigma_y A\tan\theta)\cos\theta - (\tau_{yx} A\tan\theta)\sin\theta = 0$$

Using $\tau_{xy} = \tau_{yx}$ and simplifying gives :

$$\sigma_{x1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x1y1} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Using the following trigonometric identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

gives the transformation equations for plane stress:

$$\boxed{\begin{aligned}\sigma_{x1} &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x1y1} &= -\frac{\left(\sigma_x-\sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}}$$

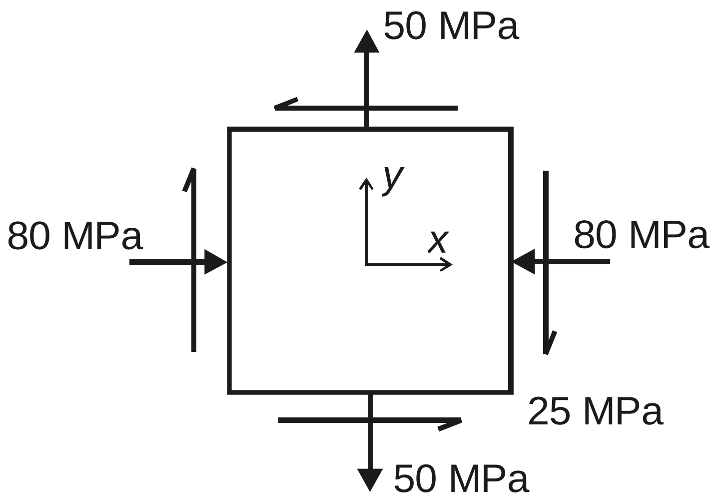
For stresses on the y_1 face, substitute $\theta + 90^\circ$ for θ:

$$\boxed{\sigma_{y1} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

Summing the expressions for x_1 and y_1 gives:

$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$ Can be used to find σ_{y1}, instead of eqn above.

Example: The state of plane stress at a point is represented by the stress element below. Determine the stresses acting on an element oriented 30° clockwise with respect to the original element.



Define the stresses in terms of the established sign convention:

σ_x = -80 MPa σ_y = 50 MPa

τ_{xy} = -25 MPa

We need to find σ_{x1}, σ_{y1}, and τ_{x1y1} when θ = -30°.

Substitute numerical values into the transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

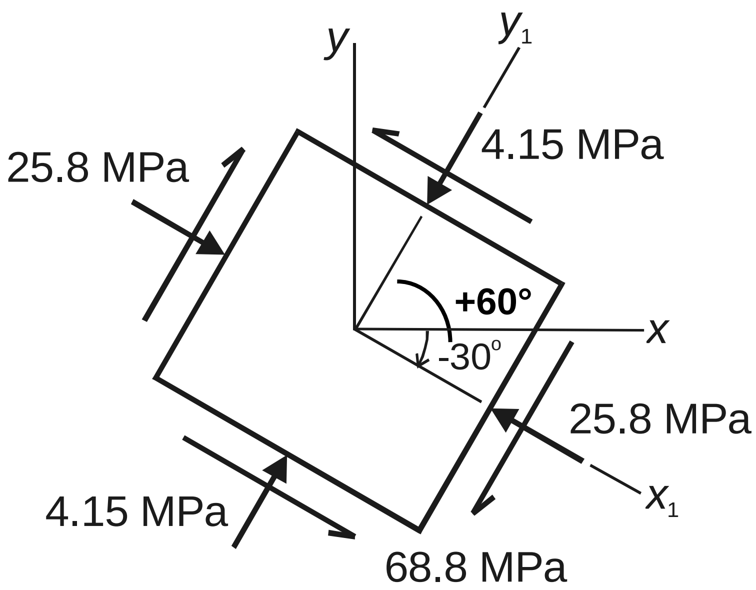
$$\sigma_{x1} = \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(-30^\circ) + (-25)\sin 2(-30^\circ) = -25.9 \text{ MPa}$$

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{y1} = \frac{-80+50}{2} - \frac{-80-50}{2}\cos 2(-30°) - (-25)\sin 2(-30°) = -4.15 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x1y1} = -\frac{(-80-50)}{2}\sin 2(-30°) + (-25)\cos 2(-30°) = -68.8 \text{ MPa}$$



Note that σ_{y1} could also be obtained (a) by substituting +60° into the equation for σ_{x1} or (b) by using the equation $\sigma_x + \sigma_y = \sigma_{x1} + \sigma_{y1}$

Principal Stresses

The maximum and minimum normal stresses (σ_1 and σ_2) are known as the **principal stresses**. To find the principal stresses, we must differentiate the transformation equations.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_{x1}}{d\theta} = \frac{\sigma_x - \sigma_y}{2}\left(-2\sin 2\theta\right) + \tau_{xy}\left(2\cos 2\theta\right) = 0$$

$$\frac{d\sigma_{x1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\boxed{\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}}$$

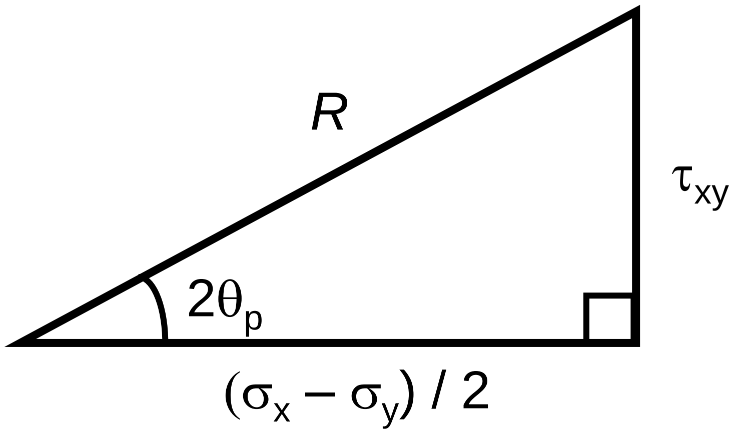
θ_p are **principal angles** associated with the principal stresses

There are two values of $2\theta_p$ in the range 0-360°, with values differing by 180°.
There are two values of θ_p in the range 0-180°, with values differing by 90°.
So, the planes on which the principal stresses act are mutually perpendicular.

We can now solve for the principal stresses by substituting for θ_p in the stress transformation equation for σ_{x1}. This tells us which principal stress is associated with which principal angle.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$



$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right)$$

Substituting for *R* and re-arranging gives the larger of the two principal stresses:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

To find the smaller principal stress, use $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$.

$$\sigma_2 = \sigma_x + \sigma_y - \sigma_1 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

These equations can be combined to give:

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}$$

Principal stresses

To find out which principal stress goes with which principal angle, we could use the equations for sin θ_p and cos θ_p or for σ_{x1}.

The planes on which the principal stresses act are called the **principal planes**. What shear stresses act on the principal planes?

Compare the equations for $\tau_{x1y1} = 0$ and $d\sigma_{x1}/d\theta = 0$

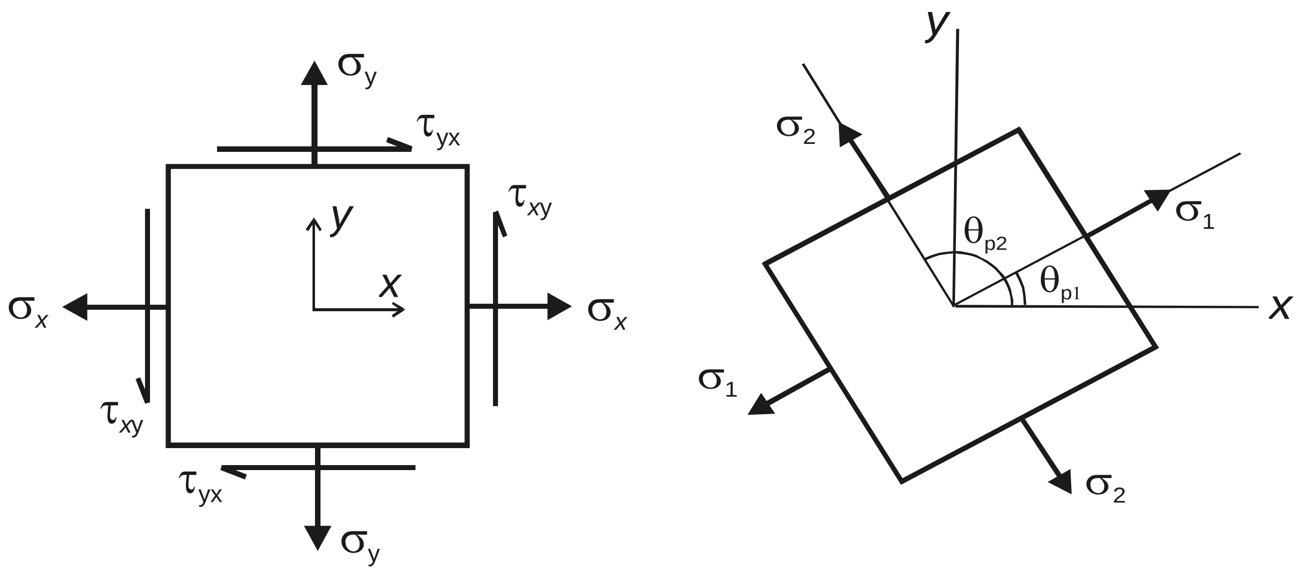
$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = 0$$

$$-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\frac{d\sigma_{x1}}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Solving either equation gives the same expression for tan $2\theta_p$

Hence, **the shear stresses are zero on the principal planes**.



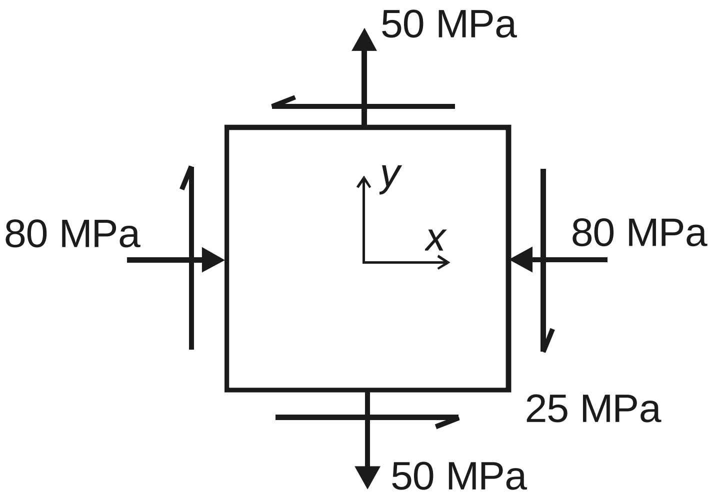
Principal Stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Angles defining the **Principal Planes**

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Example: The state of plane stress at a point is represented by the stress element below. Determine the principal stresses and draw the corresponding stress element.



Define the stresses in terms of the established sign convention:

σ_x = -80 MPa σ_y = 50 MPa

τ_{xy} = -25 MPa

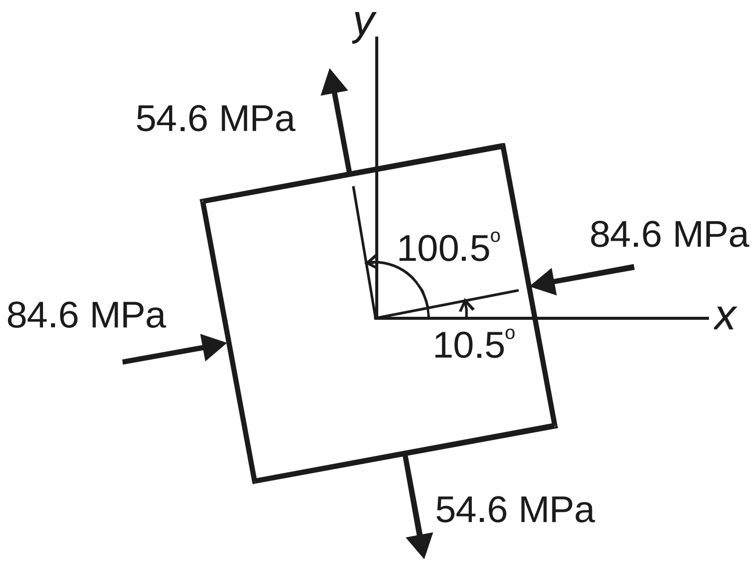
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{1,2} = \frac{-80+50}{2} \pm \sqrt{\left(\frac{-80-50}{2}\right)^2 + (-25)^2} = -15 \pm 69.6$$

$$\sigma_1 = 54.6 \text{ MPa} \qquad \sigma_2 = -84.6 \text{ MPa}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_p = \frac{2(-25)}{-80-50} = 0.3846$$



$$2\theta_p = 21.0° \text{ and } 21.0 + 180°$$

$$\theta_p = 10.5°, 100.5°$$

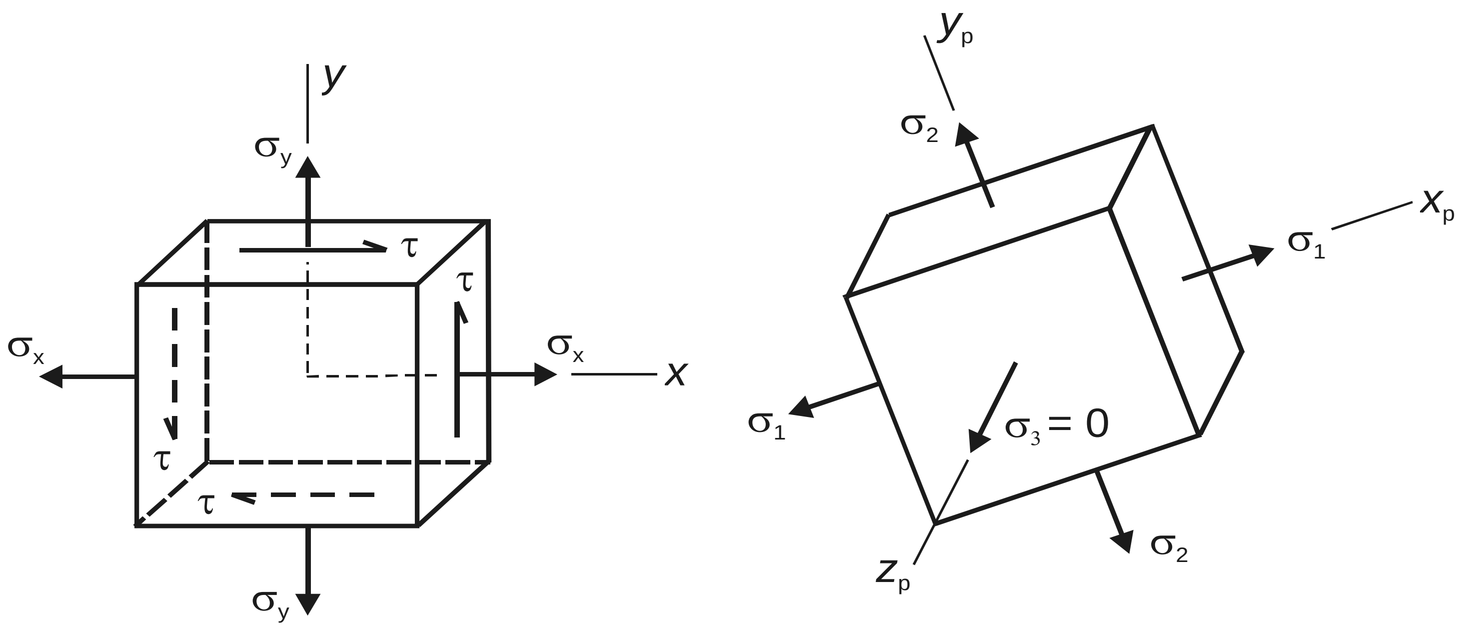
But we must check which angle goes with which principal stress.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x1} = \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(10.5°) + (-25)\sin 2(10.5°) = -84.6 \text{ MPa}$$

σ_1 = 54.6 MPa with θ_{p1} = 100.5°
σ_2 = -84.6 MPa with θ_{p2} = 10.5°

The two principal stresses determined so far are the principal stresses in the *xy* plane. But … remember that the stress element is 3D, so there are always **three principal stresses**.



σ_x, σ_y, $\tau_{xy} = \tau_{yx} = \tau$

σ_1, σ_2, $\sigma_3 = 0$

Usually we take $\sigma_1 > \sigma_2 > \sigma_3$. Since principal stresses can be compressive as well as tensile, σ_3 could be a negative (compressive) stress, rather than the zero stress.

Maximum Shear Stress

To find the maximum shear stress, we must differentiate the transformation equation for shear.

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_{x1y1}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)$$

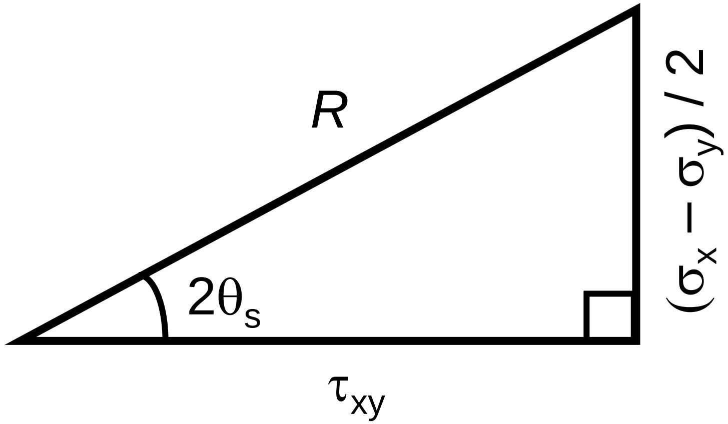
There are two values of $2\theta_s$ in the range 0-360°, with values differing by 180°. There are two values of θ_s in the range 0-180°, with values differing by 90°. So, the planes on which the maximum shear stresses act are mutually perpendicular.

Because shear stresses on perpendicular planes have equal magnitudes, the maximum positive and negative shear stresses differ only in sign.

We can now solve for the maximum shear stress by substituting for θ_s in the stress transformation equation for τ_{x1y1}.

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



$$R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}$$

$$\cos 2\theta_s = \frac{\tau_{xy}}{R}$$

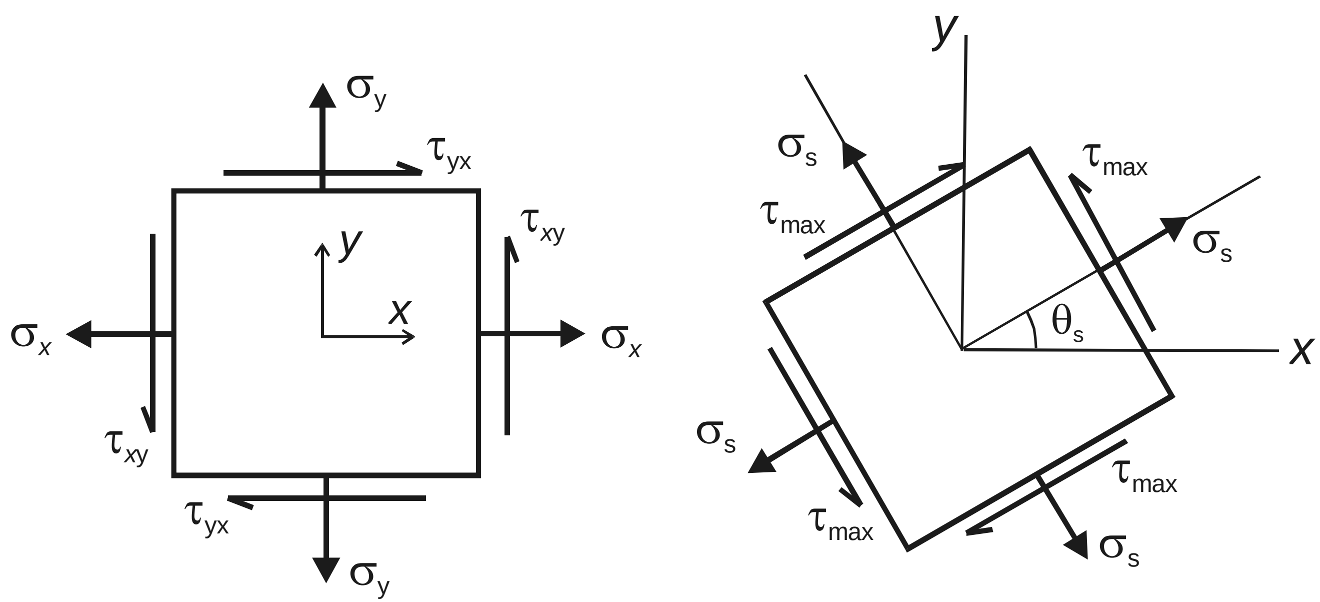
$$\sin 2\theta_s = -\frac{\sigma_x - \sigma_y}{2R}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} \qquad \tau_{\min} = -\tau_{\max}$$

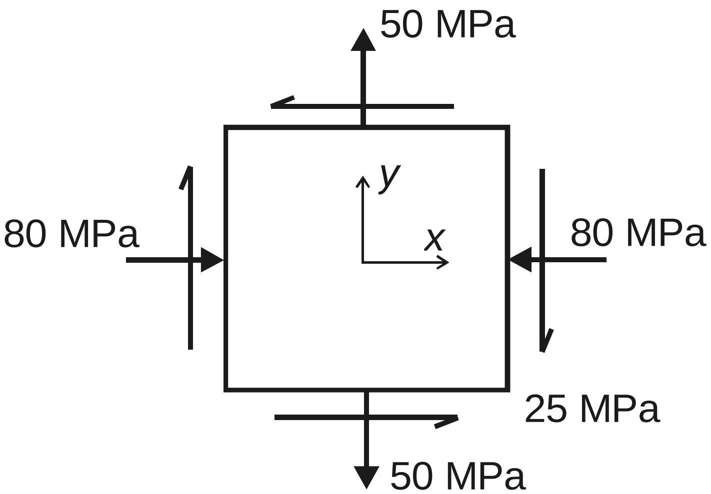
Use equations for sin θ_s and cos θ_s or τ_{x1y1} to find out which face has the positive shear stress and which the negative.

What normal stresses act on the planes with maximum shear stress? Substitute for θ_s in the equations for σ_{x1} and σ_{y1} to get

$$\sigma_{x1} = \sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} = \sigma_s$$



Example: The state of plane stress at a point is represented by the stress element below. Determine the maximum shear stresses and draw the corresponding stress element.



Define the stresses in terms of the established sign convention:

σ_x = -80 MPa σ_y = 50 MPa

τ_{xy} = -25 MPa

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{-80-50}{2}\right)^2 + (-25)^2} = 69.6 \text{ MPa}$$

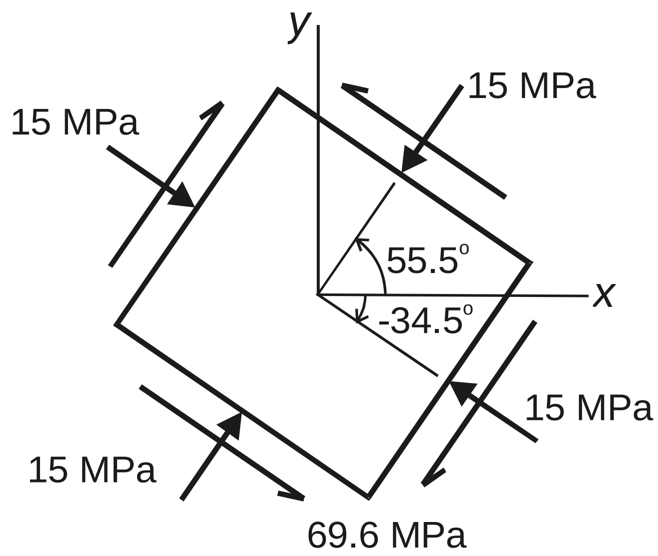
$$\sigma_s = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_s = \frac{-80+50}{2} = -15 \text{ MPa}$$

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) = -\left(\frac{-80-50}{2(-25)}\right) = -2.6$$

$$2\theta_s = -69.0^\circ \text{ and } -69.0 + 180^\circ$$

$$\theta_s = -34.5^\circ, 55.5^\circ$$



But we must check which angle goes with which shear stress.

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x1y1} = -\frac{(-80-50)}{2}\sin 2(-34.5) + (-25)\cos 2(-34.5) = -69.6\text{ MPa}$$

τ_{max} = 69.6 MPa with θ_{smax} = 55.5°
τ_{min} = -69.6 MPa with θ_{smin} = -34.5°

Finally, we can ask how the principal stresses and maximum shear stresses are related and how the principal angles and maximum shear angles are related.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 - \sigma_2 = 2\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_1 - \sigma_2 = 2\tau_{\max}$$

$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}}$$

$$\tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) \qquad \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_s = \frac{-1}{\tan 2\theta_p} = -\cot 2\theta_p$$

$$\tan 2\theta_s + \cot 2\theta_p = 0$$

$$\frac{\sin 2\theta_s}{\cos 2\theta_s} + \frac{\cos 2\theta_p}{\sin 2\theta_p} = 0$$

$$\sin 2\theta_s \sin 2\theta_p + \cos 2\theta_s \cos 2\theta_p = 0$$

$$\cos\left(2\theta_s - 2\theta_p\right) = 0$$

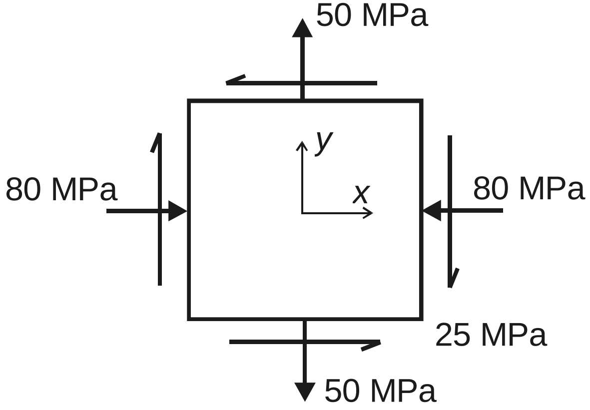
$$2\theta_s - 2\theta_p = \pm 90°$$

$$\theta_s - \theta_p = \pm 45°$$

$$\boxed{\theta_s = \theta_p \pm 45°}$$

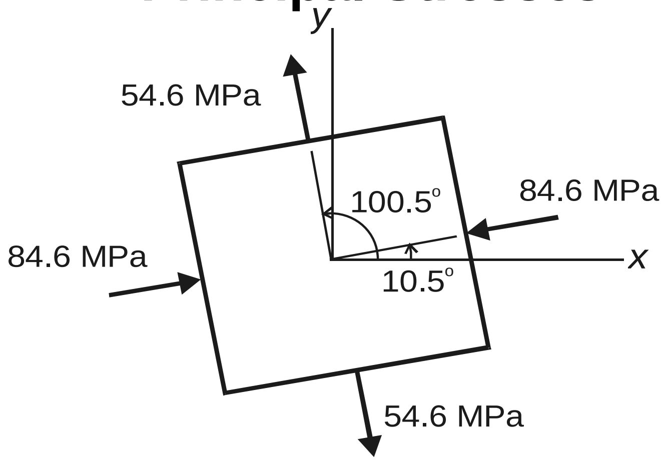
So, the planes of maximum shear stress (θ_s) occur at 45° to the principal planes (θ_p).

Original Problem



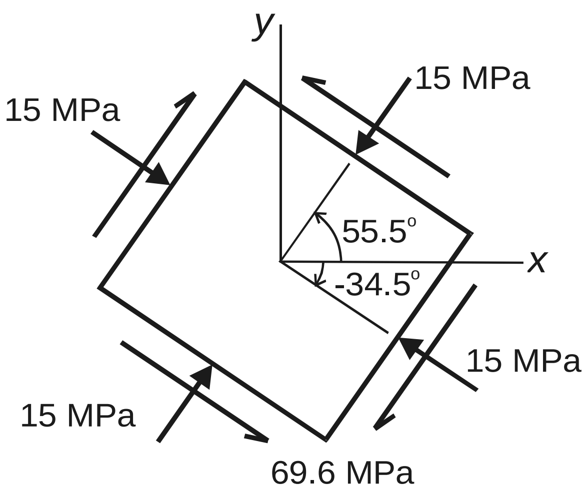
$\sigma_x = -80$, $\sigma_y = 50$, $\tau_{xy} = 25$

Principal Stresses



$\sigma_1 = 54.6$, $\sigma_2 = 0$, $\sigma_3 = -84.6$

Maximum Shear



$\tau_{max} = 69.6$, $\sigma_s = -15$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{\max} = \frac{54.6 - (-84.6)}{2}$$

$$\tau_{\max} = 69.6 \text{ MPa}$$

$$\theta_s = \theta_p \pm 45°$$

$$\theta_s = 10.5° \pm 45°$$

$$\theta_s = -34.5°, 55.5°$$