

LU Decomposition

Introduction

- Another way of solving a system of equations is by using a factorization technique for matrices called **LU decomposition**.
- This factorization involves two matrices, one lower triangular matrix and one upper triangular matrix.
- *LU* factorization methods separate the time-consuming elimination of the matrix $[A]$ from the manipulations of the right-hand-side $[b]$.
- Once $[A]$ has been factored (or decomposed), multiple right-hand-side vectors can be evaluated in an efficient manner.

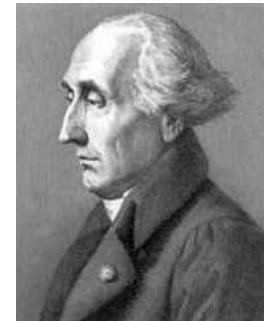
LU Decomposition

LU decomposition was originally derived as a decomposition of quadratic and bilinear forms. Lagrange, in the very first paper in his collected works (1759) derives the algorithm we call Gaussian elimination. Later Turing introduced the LU decomposition of a matrix in 1948 that is used to solve the system of linear equation.

Let A be a $m \times m$ with nonsingular square matrix. Then there exists two matrices L and U such that, where L is a lower triangular matrix and U is an upper triangular matrix.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ 0 & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{mm} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mm} \end{bmatrix}$$

Where, $A = LU$



J-L Lagrange
(1736 -1813)



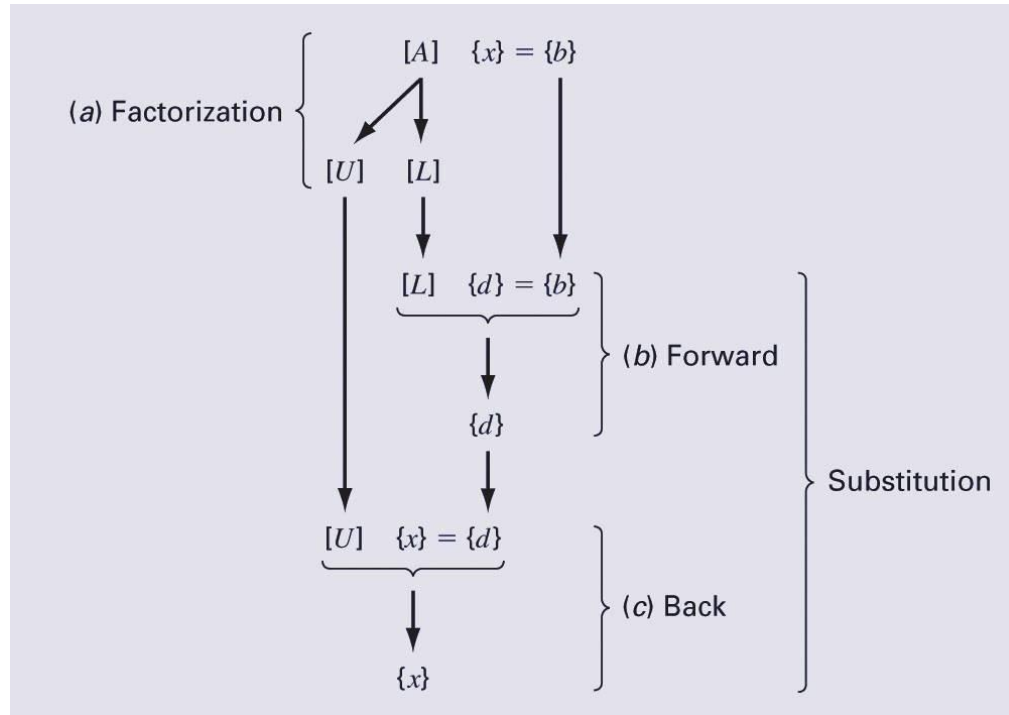
A. M. Turing
(1912-1954)

LU Factorization

- LU factorization involves two steps:

- Factorization to decompose the $[A]$ matrix into a product of a lower triangular matrix $[L]$ and an upper triangular matrix $[U]$. $[L]$ has 1 for each entry on the diagonal.
- Substitution to solve for $\{x\}$

- Gauss elimination can be implemented using LU factorization



LU Decomposition

LU Decomposition is another method to solve a set of simultaneous linear equations

Which is better, Gauss Elimination or LU Decomposition?

To answer this, a closer look at LU decomposition is needed.

LU Decomposition

Method

For most non-singular matrix $[A]$ that one could conduct Naïve Gauss Elimination forward elimination steps, one can always write it as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

How does LU Decomposition work?

If solving a set of linear equations

$$[A][X] = [C]$$

If $[A] = [L][U]$ then

$$[L][U][X] = [C]$$

Multiply by

$$[L]^{-1}$$

Which gives

$$[L]^{-1}[L][U][X] = [L]^{-1}[C]$$

Remember $[L]^{-1}[L] = [I]$ which
leads to

$$[I][U][X] = [L]^{-1}[C]$$

$$[U][X] = [L]^{-1}[C]$$

Now, if $[I][U] = [U]$ then

$$[L]^{-1}[C] = [Z]$$

Now, let

$$[L][Z] = [C] \quad (1)$$

Which ends with

$$[U][X] = [Z] \quad (2)$$

and

LU Decomposition

How can this be used?

Given $[A][X] = [C]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [C]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

When is LU Decomposition better than Gaussian Elimination?

$$\text{To solve } [A][X] = [B]$$

Table. Time taken by methods

Gaussian Elimination	LU Decomposition
$T \left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3} \right)$	$T \left(\frac{8n^3}{3} + 12n^2 + \frac{4n}{3} \right)$

where T = clock cycle time and n = size of the matrix

So both methods are equally efficient.

To find inverse of $[A]$

Time taken by Gaussian Elimination

$$\begin{aligned} &= n \text{ CT}|_{FE} + \text{CT}|_{BS} \\ &= T \left(\frac{8n^4}{3} + 12n^3 + \frac{4n^2}{3} \right) \end{aligned}$$

Time taken by LU Decomposition

$$\begin{aligned} &= \text{CT}|_{LU} + n \times \text{CT}|_{FS} + n \times \text{CT}|_{BS} \\ &= T \left(\frac{32n^3}{3} + 12n^2 + \frac{20n}{3} \right) \end{aligned}$$

Table 1 Comparing computational times of finding inverse of a matrix using LU decomposition and Gaussian elimination.

n	10	100	1000	10000
$\text{CT} _{\text{inverse GE}} / \text{CT} _{\text{inverse LU}}$	3.28	25.83	250.8	2501

Method: $[A]$ Decompose to $[L]$ and $[U]$

$$A = L U = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$[U]$ is the same as the coefficient matrix at the end of the forward elimination step.

$[L]$ is obtained using the *multipliers* that were used in the forward elimination process

Finding the $[U]$ matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\text{Step 1: } \frac{64}{25} = 2.56; \quad \text{Row2} - \text{Row1} \cdot 2.56 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\frac{144}{25} = 5.76; \quad \text{Row3} - \text{Row1} \cdot 5.76 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

$$\text{Matrix after Step 1: } \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\text{Step 2: } \frac{-16.8}{-4.8} = 3.5; \quad \text{Row3} - \text{Row2} \cdot 3.5 = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$U = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the $[L]$ matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \quad \ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$
$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second
step of forward
elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does $[L][U] = [A]$?

$$L U = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} = ?$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the $[L]$ and $[U]$ matrices

$$A = L U = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Links

<https://www.geeksforgeeks.org/doolittle-algorithm-lu-decomposition/>

Doolittle Algorithm :

It is always possible to factor a square matrix into a lower triangular matrix and an upper triangular matrix. That is, $[A] = [L][U]$

Doolittle's method provides an alternative way to factor A into an LU decomposition without going through the hassle of [Gaussian Elimination](#).

<https://epxx.co/artigos/ludecomp.html>

Example

$$\text{Set } [L][Z] = [C]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\text{Solve for } [Z]$$

$$z_1 = 10$$

$$2.56z_1 + z_2 = 177.2$$

$$5.76z_1 + 3.5z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for $[Z]$

$$z_1 = 106.8$$

$$\begin{aligned} z_2 &= 177.2 - 2.56z_1 \\ &= 177.2 - 2.56(106.8) \\ &= -96.2 \end{aligned}$$

$$\begin{aligned} z_3 &= 279.2 - 5.76z_1 - 3.5z_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.21) \\ &= 0.735 \end{aligned}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example

$$\text{Set } [U][X] = [Z] \quad \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$\begin{aligned} 25a_1 + 5a_2 + a_3 &= 106.8 \\ -4.8a_2 - 1.56a_3 &= -96.21 \\ 0.7a_3 &= 0.735 \end{aligned}$$

Example

From the 3rd equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$\begin{aligned} a_1 &= \frac{106.8 - 5a_2 - a_3}{25} \\ &= \frac{106.8 - 5(19.70) - 1.050}{25} \\ &= 0.2900 \end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Finding the inverse of a square matrix

The inverse $[B]$ of a square matrix $[A]$ is defined as

$$[A][B] = [I] = [B][A]$$

Finding the inverse of a square matrix

How can LU Decomposition be used to find the inverse?

Assume the first column of $[B]$ to be $[b_{11} \ b_{12} \ \dots \ b_{n1}]^T$

Using this and the definition of matrix multiplication

First column of $[B]$

$$A \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Second column of $[B]$

$$A \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The remaining columns in $[B]$ can be found in the same manner

Example: Inverse of a Matrix

Find the inverse of a square matrix $[A]$

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Using the decomposition procedure, the $[L]$ and $[U]$ matrices are found to be

$$A = L U = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving for the each column of $[B]$ requires two steps

1) Solve $[L] [Z] = [C]$ for $[Z]$

2) Solve $[U] [X] = [Z]$ for $[X]$

$$\text{Step 1: } L Z = C \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This generates the equations:

$$z_1 = 1$$

$$2.56z_1 + z_2 = 0$$

$$5.76z_1 + 3.5z_2 + z_3 = 0$$

Example: Inverse of a Matrix

Solving for [Z]

$$z_1 = 1$$

$$\begin{aligned} z_2 &= 0 - 2.56z_1 \\ &= 0 - 2.56(1) \\ &= -2.56 \end{aligned}$$

$$\begin{aligned} z_3 &= 0 - 5.76z_1 - 3.5z_2 \\ &= 0 - 5.76(1) - 3.5(-2.56) \\ &= 3.2 \end{aligned}$$

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

Example: Inverse of a Matrix

Solving $[U][X] = [Z]$ for $[X]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$25b_{11} + 5b_{21} + b_{31} = 1$$

$$-4.8b_{21} - 1.56b_{31} = -2.56$$

$$0.7b_{31} = 3.2$$

Example: Inverse of a Matrix

Using Backward Substitution

$$b_{31} = \frac{3.2}{0.7} = 4.571$$

$$\begin{aligned} b_{21} &= \frac{-2.56 + 1.560b_{31}}{-4.8} \\ &= \frac{-2.56 + 1.560(4.571)}{-4.8} = -0.9524 \end{aligned}$$

$$\begin{aligned} b_{11} &= \frac{1 - 5b_{21} - b_{31}}{25} \\ &= \frac{1 - 5(-0.9524) - 4.571}{25} = 0.04762 \end{aligned}$$

So the first column of the inverse of $[A]$ is:

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Example: Inverse of a Matrix

Repeating for the second and third columns of the inverse

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example: Inverse of a Matrix

The inverse of $[A]$ is

$$A^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

To check your work do the following operation

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$

Cholesky Decomposition

Cholesky died from wounds received on the battle field on 31 August 1918 at 5 o'clock in the morning in the North of France. After his death one of his fellow officers, Commandant Benoit, published Cholesky's method of computing solutions to the normal equations for some least squares data fitting problems published in the *Bulletin géodesique* in 1924. Which is known as Cholesky Decomposition

Cholesky Decomposition: If A is a real, symmetric and positive definite matrix then there exists a unique lower triangular matrix L with positive diagonal element such that

$$A = LL^T$$



Andre-Louis Cholesky

1875-1918

Cholesky Factorization

- Symmetric systems occur commonly in both mathematical and engineering/science problem contexts, and there are special solution techniques available for such systems.
- The *Cholesky factorization* is one of the most popular of these techniques, and is based on the fact that a symmetric matrix can be decomposed as $[A] = [U]^T[U]$, where T stands for transpose.
- The rest of the process is similar to *LU* decomposition and Gauss elimination, except only one matrix, $[U]$, needs to be stored.