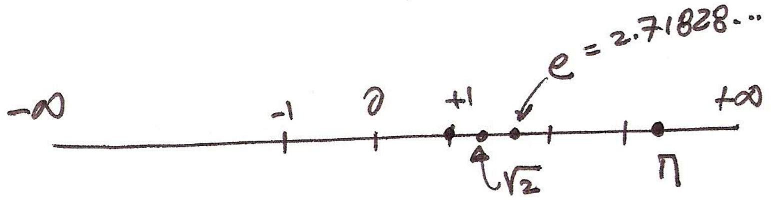
Μαθηματική Θεωρία

(Φροντιστήριο)
(1)

2015-2016

Μαθηματική Θεωρία



$\mathbb{R}$ (πραγματικοί αριθμοί)

Πράξεις $a \pm b$, ab, a/b, $a+0=a$, $a \cdot 1 = a$, $a^{-1} = 1/a$

$x \cdot a = 1$, $x = 1/a = a^{-1}$ | $x = -a$, $x + a = 0$

$a > b$, $b > a$, **ανισότητες**

$a - b > 0 \Rightarrow a > b$

$$|a| = \begin{cases} a & a > 0 \\ -a & a < 0 \\ 0 & a = 0 \end{cases}$$

απόλυτη τιμή

Ρίζες – Δυνάμεις – εκθέτες

$$\begin{cases} a^{p/q} = \sqrt[q]{a^p} \\ a^{-q} = \dfrac{1}{a^q} \end{cases}$$

Λογάριθμοι $a^P = N \Rightarrow \boxed{P = \log_a N}$

$$\log_a(MN) = \log_a M + \log_a N, \quad \log_a \frac{M}{N} = \log_a M - \log_a N$$

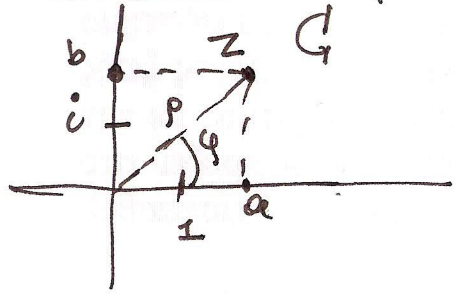
$$\log_a M^\tau = \tau \log_a M,$$

$$\log_a X = \log_b X \log_a b$$

Μιγαδικοί αριθμοί $\boxed{i^2 = -1}$

(Euler)

$$e^{i\varphi} = \cos\varphi + i \sin\varphi$$



$$z = a + ib = \rho(\cos\varphi + i \sin\varphi)$$

$$z_1 z_2 = \rho_1 \rho_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}[\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$$

$$z^n = \rho^n [\cos(n\varphi) + i \sin(n\varphi)]$$

$$z^{1/n} = [\rho(\cos\varphi + i\sin\varphi)]^{1/n} = \rho^{1/n} \left\{ \cos\frac{\varphi + 2k\pi}{n} + i \sin\frac{\varphi + 2k\pi}{n} \right\}$$

$k = 0, 1, 2, \ldots n-1$

Μαθηματική επαγωγή $P(n) = \text{αληθές} \ (\forall n)$

$P(n=1) = \text{αληθές}$ (απόδειξη)

$P(n=k) = \text{αληθές}$ (υπόθεση) $\Rightarrow$ $P(n=k+1) = \text{αληθές}$ (απόδειξη)

$P(n) = \text{αληθές} \quad \forall\, n = 1, 2, \dots \infty$

Σημεία – Σύνολα – όρια

R, C (συνεχές) (c)

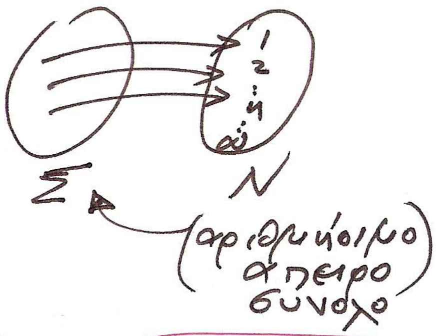
$N \sim Z \sim Q$ (αριθμήσιμα) $\aleph_0$

$$\boxed{N \subset Z \subset Q \subset R \subset C}$$

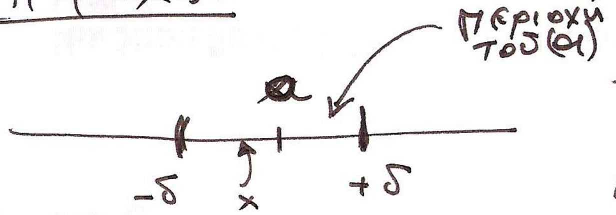
$\Sigma = \{\sigma_1, \sigma_2 \dots \sigma_n \dots\}$

$\sigma_n \leftrightarrow n$

(αριθμήσιμο απειροσύνολο) ($\aleph_0$)



Περιοχή



$$\boxed{\begin{array}{l} x < \alpha + \delta \\ x > \alpha - \delta \end{array}}$$

$$\boxed{|\alpha - \delta < x < \alpha + \delta|} \longleftrightarrow |x - \alpha| < \delta$$

Θεώρημα Weierstrass-Bolzano

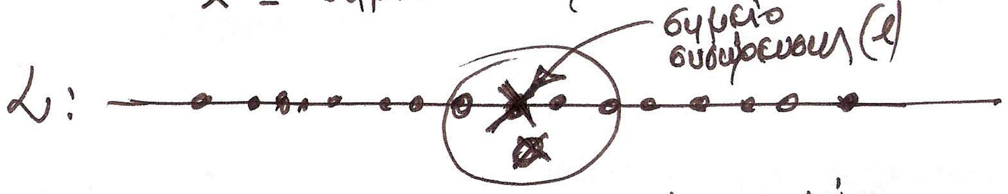
Κάθε φραγμένο απειροσύνολο έχει ένα τουλάχιστον οριακό σημείο

$\boxed{\text{οριακό σημείο – σημείο συσσώρευσης}}$

L = σύνολο αριθμών (σημείων)

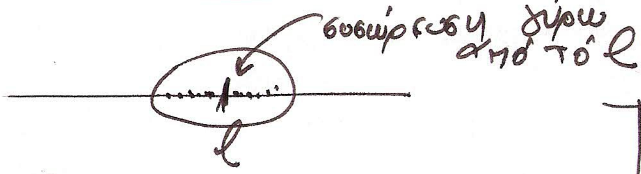
x = σημείο συσσώρευσης

$\boxed{\ell = \text{οριακό σημείο του } L}$



$\delta \to 0$

$|x - \ell| < \delta$



Κλειστό σύνολο: περιέχει όλα τα οριακά του σημεία

φράγμα (άνω, κάτω)

$m \leq x \leq M \quad \forall x \in L$

$\boxed{\text{φραγμένο σύνολο } L}$

Ακολουθίες - Σειρές αριθμών

$$f(n) = u_n = \{u_1, u_2, \dots u_n \dots\}$$ ακολουθία

όριο ακολουθίας

$u_1, u_2, u_3 \dots u_n \dots$

$$l = \underset{n\to\infty}{\text{όριο}}\, u_n \qquad \boxed{\forall \varepsilon>0,\ |u_n - l| < \varepsilon \quad \forall\, n > n(\varepsilon)}$$

$$\text{ορ}\, a_n = A, \quad \text{ορ}\, b_n = B \qquad \text{ορ}\,(a_n + b_n) = A + B = \text{ορ}\, a_n + \text{ορ}\, b_n$$

$$\text{ορ}\,(a_n \pm b_n) = \text{ορ}\, a_n \pm \text{ορ}\, b_n = A \pm B$$

$$\text{ορ}\,(a_n b_n) = (\text{ορ}\, a_n)(\text{ορ}\, b_n) = A \cdot B$$

$$\text{ορ}\, \frac{a_n}{b_n} = \frac{\text{ορ}\, a_n}{\text{ορ}\, b_n}, \quad \text{ορ}\, b_n = B \neq 0$$

$$\text{ορ}\left(a_n^{p}\right) = (\text{ορ}\, a_n)^p = A^p, \quad \text{ορ}\left(p^{a_n}\right) = p^{\text{ορ}\, a_n} = p^A$$

Σειρές

$$\underbrace{\{u_1, u_2, \dots u_n \dots\}}_{\text{ακολουθία}} \Longrightarrow \underbrace{\{s_1, s_2, \dots s_n, \dots\}}_{\text{σειρά}}$$

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \quad s_3 = u_1 + u_2 + u_3 \quad \dots\ s_n = \sum_{i=1}^{n} u_i \ \dots$$

$$\text{ορ}\, S_n = S \quad (\text{όριο σειράς})$$

Συναρτήσεις

$$y = f(x): \quad x \to y, \quad R \xrightarrow{f} R$$

Πολυωνυμικές: $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$

Αλγεβρικές: $y = f(x)$:

$$P_0(x) y^n + P_1(x) y^{n-1} + \dots + P_{n-1}(x) y + P_n(x) = 0$$

$\boxed{\text{Υπερβατικές}} \rightarrow$ $y = f(x)$: όχι αλγεβρικές

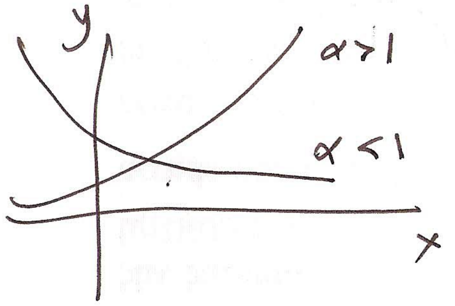
$\boxed{\text{Είδη υπερβατικών συναρτήσεων}}$

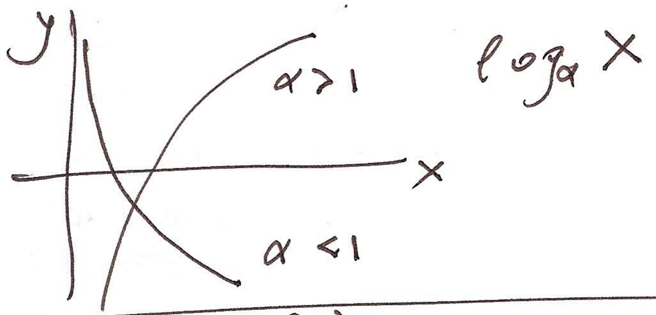
$\boxed{\text{Εκθετική συνάρτηση}}$ $\underline{y = f(x) \equiv a^x}$ $\quad a \neq 0, 1$

$\boxed{\text{Λογαριθμική συνάρτηση}}$ $\underline{y = f(x) \equiv \log_a x}$ $\quad a \neq 0, 1$

$y = f(x) \equiv \ln x = \log_e x$, $\underline{e = 2.71828\ldots}$

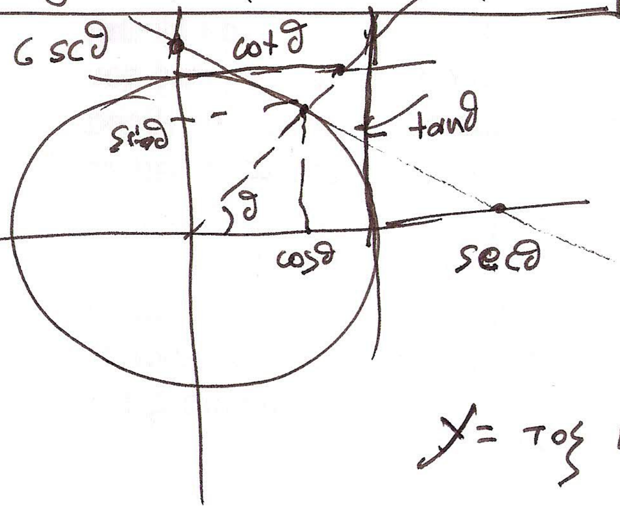
a^x

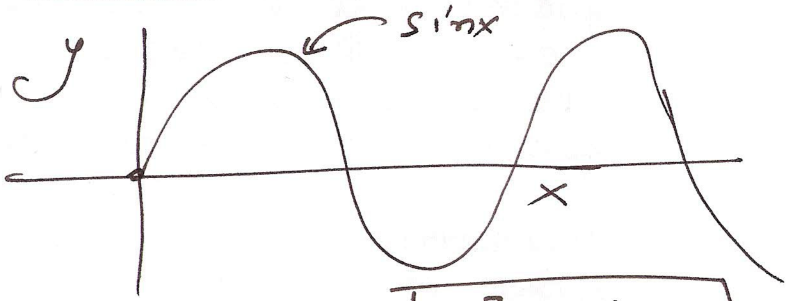




$\boxed{\text{Τριγωνομετρική συνάρτηση}}$

$y = f(x)$:   sin x, cos x, tan x, cot x   scc x, sec x





$y = \text{τοξ ημ}\, x = \sin^{-1} x$ $\quad \boxed{\cos^2 x + \sin^2 x = 1}$

$\boxed{\text{Υπερβολικές συναρτήσεις}}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$. $\quad \boxed{\cosh^2 x + \sinh^2 x = 1}$

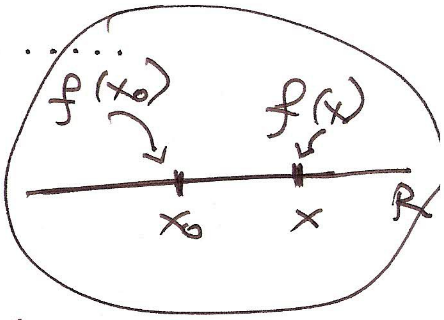
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$

Ανάπτυγμα Taylor

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \cdots$$

$$+ \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + R_n$$



$$\boxed{R_n = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}} \qquad x_0 < \xi < x$$

(υπόλοιπο Lagrange)

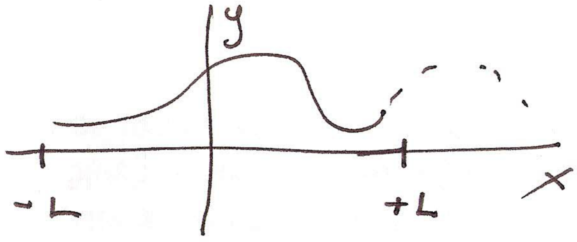
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n$$

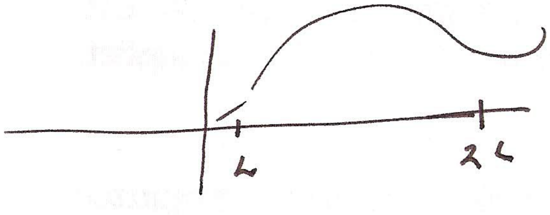
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + R_n$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} + R_n$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \cdots = \cos x + i\sin x$$

Σειρές Φουριέ (Fourier Series)





$f(x): (-L, +L)$ περιοδική

$$f(x+2L) = f(x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

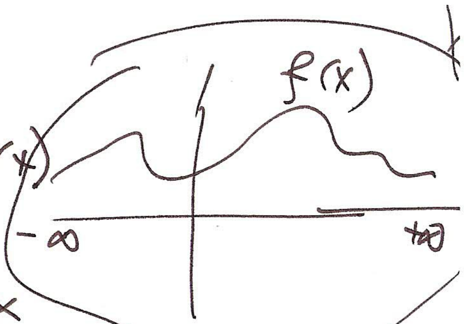
$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i(n\pi x/L)}} = c_1 e^{i(\pi x/L)} + c_2 e^{i(2\pi x/L)} + \dots$$

$$c_n = \frac{1}{2L}\int_{-L}^{+L} f(x)\, e^{-i(n\pi x/L)}\, dx$$

Ολοκλήρωμα Φουριέ για ομαλές $f(x)$



$$f(x) = \int_0^{\infty} \{A(a)\cos ax + B(a)\sin ax\}\, da$$

$$A(a) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos(ax)\,dx \,, \quad B(a) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin(ax)\,dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(a)\, e^{-iax}\, da$$

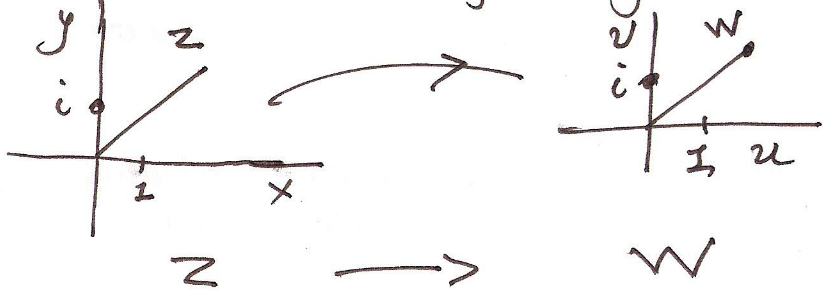
$$F(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)\, e^{iau}\, du$$

Μιγαδικές συναρτήσεις (complex functions)

$$w = f(z) = u + iv = \cancel{u(z)+iv(z)}\ u(x,y) + i\,v(x,y)$$

$$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy = u(x,y) + i\,v(x,y)$$

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$



$z \longrightarrow w$

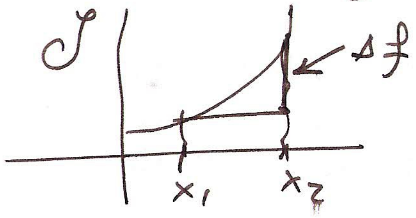
$z = x + iy \qquad w = u + iv$

Απειροστό μέγεθος

$$dx = \underset{\Delta x \to 0}{\text{ορ}} (\Delta x)$$

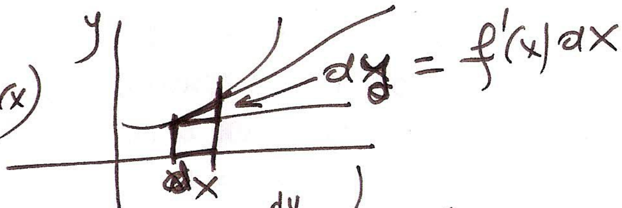
$y = f(x)$ $\quad$ ~~$\Delta x \Rightarrow \Delta y$~~ $\quad$ $\Delta x = x_2 - x_1 \Rightarrow \Delta y = y_2 - y_1$

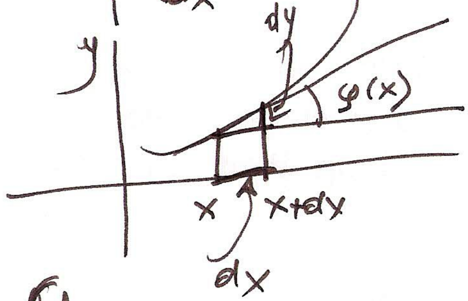
$\Delta y = \Delta f = f(x_2) - f(x_1)$



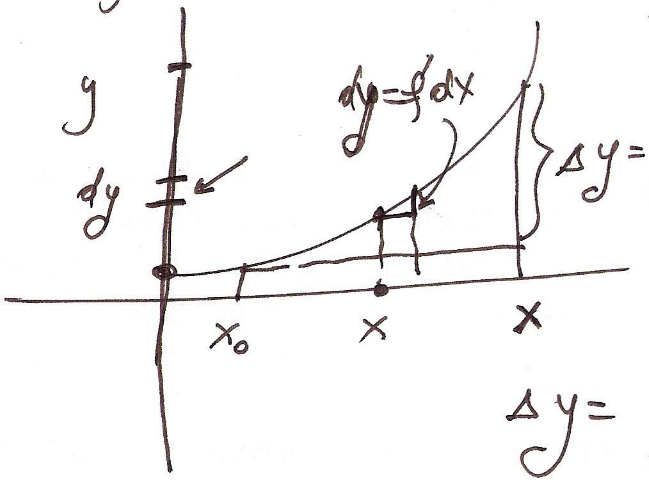
$dy = df = \text{ορ}\, \Delta f \quad (\Delta x \to 0)$

$dy = f'(x)dx \qquad f'(x) = \dfrac{dy}{dx} = \text{εφ}\,\varphi(x)$





$\Delta y = \int dy = \int df = \int f'(x)dx$

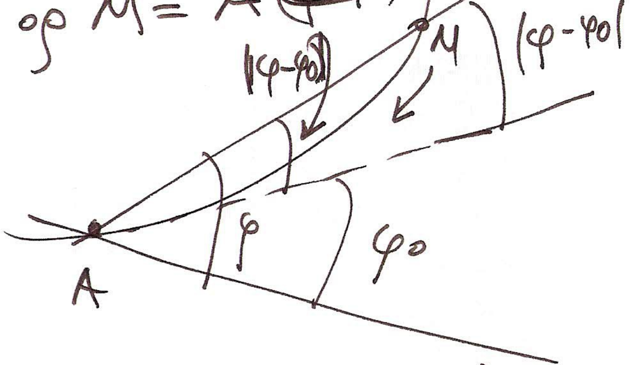


$\Delta y = \int dy = \int f'(x)dx$

$\Delta y = \int dy = \int f'(x)dx$

$\boxed{\text{όριο μεταβλητού μεγέθους}}$

$\begin{cases} M \to A \\ |M - A| \to 0 \end{cases}$ $\quad$ ή $\quad$ $\text{ορ}\, M = A$ $\quad (\varphi - \varphi_0)$



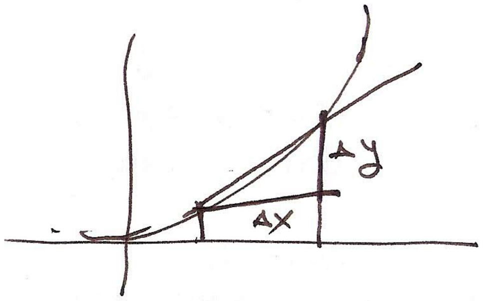
$\text{ορ}\,\varphi = \varphi_0$

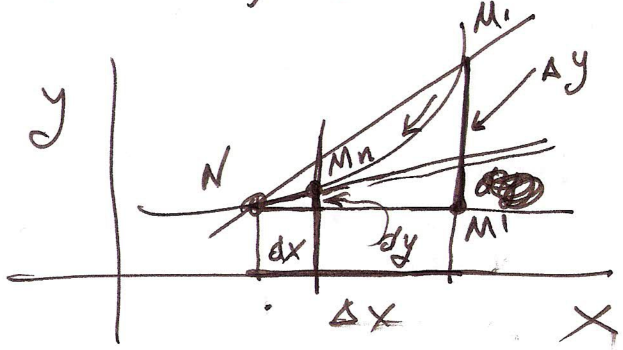
$\underset{n \to \infty}{\text{ορ}}\, S_n \to S =$ εμβαδόν

$S_n =$ εμβαδόν πολυγώνου

Τεχνικός λόγος

$y = f(x)$ $\qquad f'(x) = \text{ορ} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} =$ Τεχνικός λόγος





$\Delta x \to dx \quad , \Delta y \to dy \qquad \dfrac{\Delta y}{\Delta x} \to \dfrac{dy}{dx}$

$\underset{\Delta x \to 0}{\text{ορ}} \dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} \qquad \text{ορ} \left| \dfrac{dy}{dx} - \dfrac{\Delta y}{\Delta x} \right| = 0$

$\dfrac{\Delta y_1}{\Delta x_1} \quad \dfrac{\Delta y_2}{\Delta x_2} \cdots \dfrac{\Delta y_n}{\Delta x_n} \cdots\cdots = \dfrac{dy}{dx}$

$\Delta x_1 > \Delta x_2 > \Delta x_3 \cdots > \Delta x_n \cdots$

$\Delta(x \pm y) = \Delta x \pm \Delta y \implies d(x \pm y) = dx \pm dy$

$\Delta(x \cdot y) = (x+\Delta x)(y+\Delta y) - xy = \underline{x\Delta y + y\Delta x} + \Delta x \Delta y + (o)$

$\Delta(x \cdot y) \longrightarrow d(xy) = x dy + y dx$

$\Delta\left(\dfrac{x}{y}\right) = \dfrac{x+\Delta x}{y+\Delta y} - \dfrac{x}{y} = \dfrac{y\Delta x - x\Delta y}{y(y+\Delta y)} = \dfrac{y\Delta x - x\Delta y}{y^2 + y\Delta y}$

$\Delta\left(\dfrac{x}{y}\right) \Longrightarrow \dfrac{y dx - x dy}{y^2}$

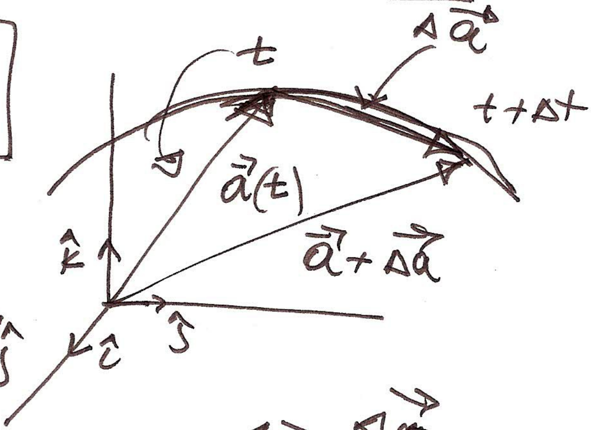
$\dfrac{d(\ldots)}{ds} = \underset{\Delta s \to 0}{\text{ορ}} \dfrac{\Delta(\ldots)}{\Delta s}$

$x = x(s) \qquad y = y(s)$

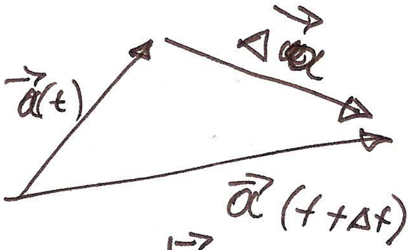
$\dfrac{d(x \pm y)}{ds} = \dfrac{dx}{ds} \pm \dfrac{dy}{ds} \qquad d\left(\dfrac{x}{y}\right) = y\dfrac{dx}{ds} - x\dfrac{dy}{ds}$

Διανυσματική συνάρτηση βαθμωτής μεταβλητής

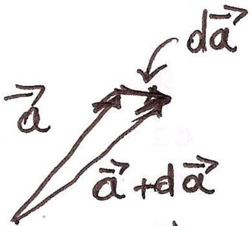
$$\boxed{\vec{a} = \vec{a}(t) = a_x(t)\,\hat{i} + a_y(t)\,\hat{j} + a_z(t)\,\hat{k}}$$



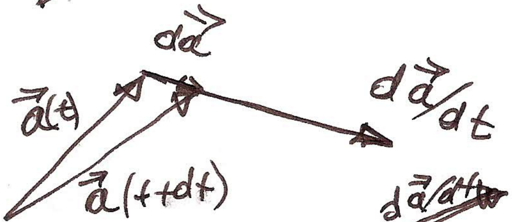
$$\Delta\vec{a} = \Delta a_x(t)\,\hat{i} + \Delta a_y\,\hat{j} + \Delta a_z\,\hat{k}$$
$$= \left[a_x(t_2) - a_x(t_1)\right]\hat{i} + \left[a_y(t_2) - a_y(t_1)\right]\hat{j}$$
$$+ \left[a_z(t_2) - a_z(t_1)\right]\hat{k}$$



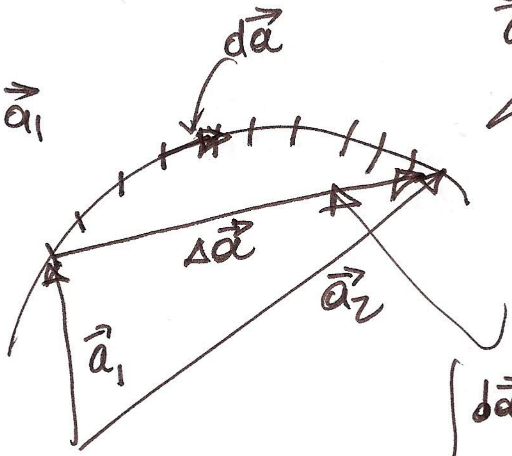
$$d\vec{a} = \text{op}\,\Delta\vec{a} = da_x\,\hat{i} + da_y\,\hat{j} + da_z\,\hat{k}$$

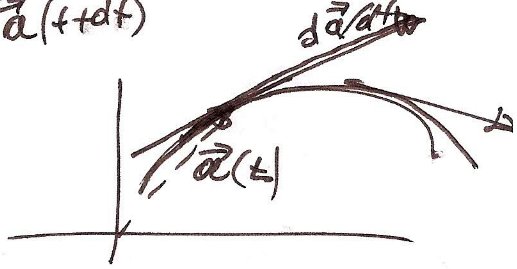


$$\boxed{\frac{d\vec{a}}{dt} = \frac{da_x}{dt}\,\hat{i} + \frac{da_y}{dt}\,\hat{j} + \frac{da_z}{dt}\,\hat{k}}$$



$$\int d\vec{a} = \Delta\vec{a} = \vec{a}_2 - \vec{a}_1$$





$$\boxed{\vec{a}_2 = \vec{a}_1 + \int d\vec{a}}$$

$$\int d\vec{a} = \Delta\vec{a}$$

$$\frac{d\vec{a}}{dt} = \vec{b}(t) \Rightarrow d\vec{a} = \vec{b}(t)\,dt \Rightarrow \int d\vec{a} = \int \vec{b}(t)\,dt$$

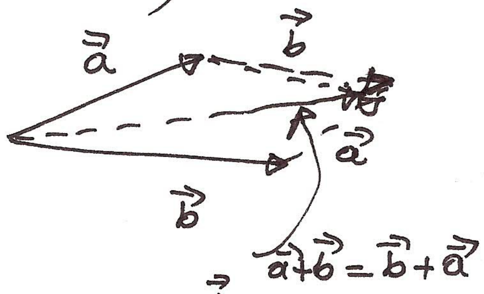
$$\Delta\vec{a} = \vec{a}_2 - \vec{a}_1 = \int d\vec{a} = \int \vec{b}\,dt = \int (b_x\,\hat{i} + b_y\,\hat{j} + b_z\,\hat{k})\,dt$$
$$= \left(\int b_x(t)\,dx\right)\hat{i} + \left(\int b_y(t)\,dx\right)\hat{j} + \left(\int b_z(t)\,dt\right)\hat{k}$$

$$\boxed{d\vec{a} = \vec{b}\cdot dt = (b_x\,dt)\,\hat{i} + (b_y\,dt)\,\hat{j} + (b_z\,dt)\,\hat{k}}$$

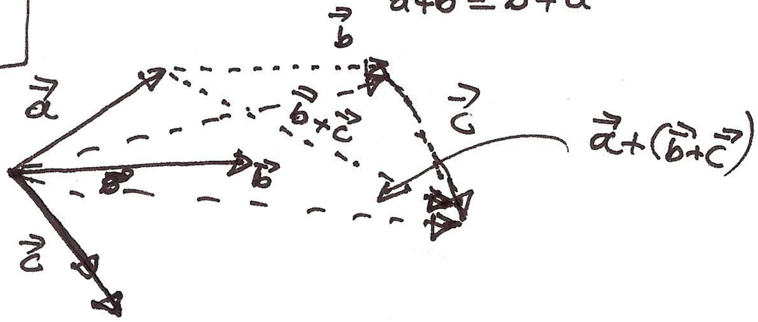
Διανυσματικός χώρος

- $\{\vec{a}, \vec{b}, \vec{c}, \dots\} \equiv \mathcal{L}$ (linear space)

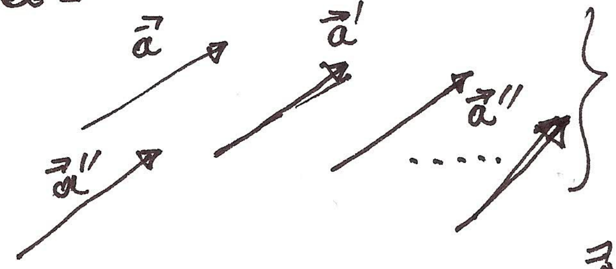
- $\boxed{\vec{a} + \vec{b} = \vec{b} + \vec{a}}$



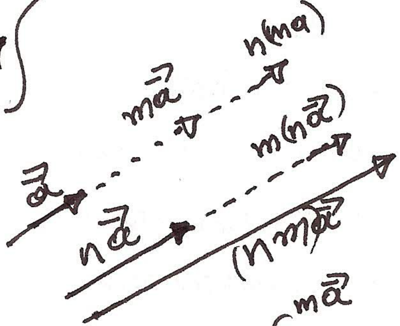
- $\boxed{\vec{a} + (\vec{b}+\vec{c}) = (\vec{a}+\vec{b}) + \vec{c} = \vec{a}+\vec{b}+\vec{c}}$



- $\vec{a} = \vec{a}' = \vec{a}'' = \dots$

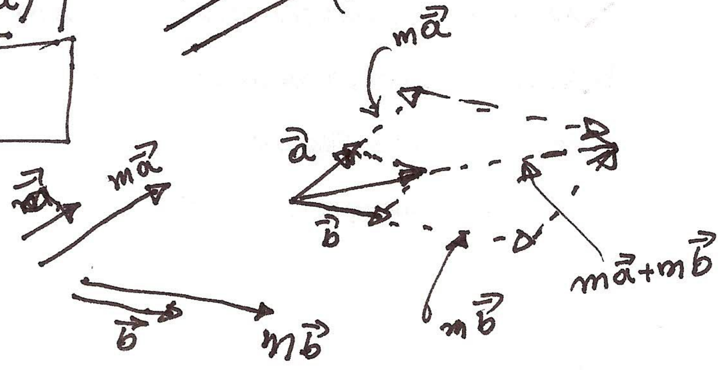


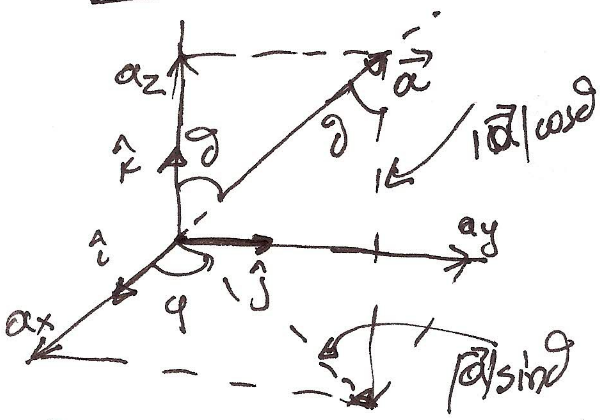
Κλάση ισοδυναμίας.



- $\boxed{n(m\vec{a}) = (nm)\vec{a} = m(n\vec{a})}$
  $\boxed{(m+n)\vec{a} = m\vec{a} + n\vec{a}}$

- $\boxed{m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}}$





$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

$(\varphi, \theta) \Rightarrow$ διεύθυνση

$|\vec{a}| =$ μέτρο

$$|\vec{a}|^2 = a_x^2 + a_y^2 + a_z^2$$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

μέτρο $|\vec{a}|$

διεύθυνση – φορά

(φ, θ)

$$\vec{a} = \vec{b} \Rightarrow a_x = b_x, \ a_y = b_y, \ a_z = b_z$$

ΔΧ

$$(|\vec{a}|, \varphi, \vartheta) \longleftrightarrow (a_x, a_y, a_z)$$

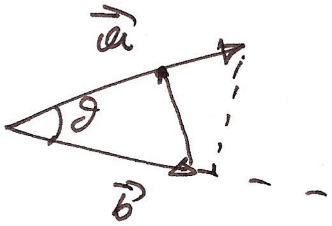
$$\left\{ \begin{aligned} a_x &= |\vec{a}| \sin\vartheta \cos\varphi \\ a_y &= |\vec{a}| \sin\vartheta \sin\varphi \\ a_z &= |\vec{a}| \cos\vartheta \end{aligned} \right\}$$

$$\boxed{|\vec{a}| = (a_x^2 + a_y^2 + a_z^2)^{1/2}}$$

<u>εσωτερικό γινόμενο</u>

$$\boxed{\begin{aligned} \vec{a}\cdot\vec{b} &= |\vec{a}||\vec{b}|\cos\vartheta \\ \vec{a}\cdot\vec{b} &= a_x b_x + a_y b_y + a_z b_z \end{aligned}}$$

$$|\vec{a}| = (\vec{a}\cdot\vec{a})^{1/2} = (a_x^2 + a_y^2 + a_z^2)^{1/2}$$



$$\hat{a}\cdot\hat{a} = 1$$

$\hat{a}$ = μοναδιαίο

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\perp\vec{b}$$

<u>εξωτερικό γινόμενο</u>

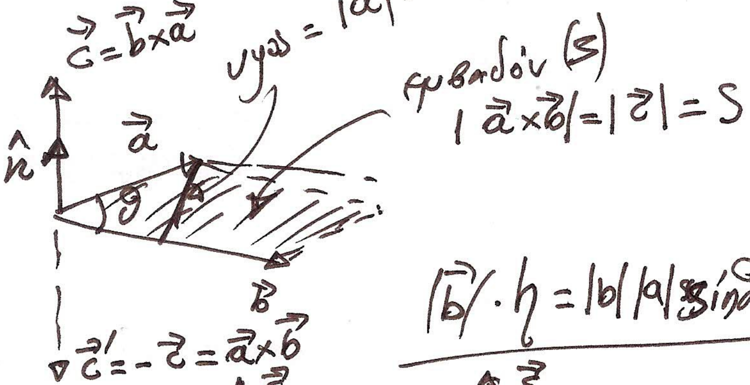
$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$$

$$\vec{c} = \hat{n}\,|\vec{a}\times\vec{b}|$$

$$|\vec{c}| = |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\vartheta$$

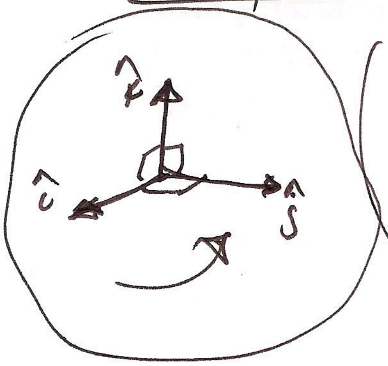
$$\underline{\vec{c} = S\,\hat{n}}$$

<u>$\hat{n}$ = μοναδιαίο</u>



$$\underline{|\vec{b}|\cdot h = |b||a|\sin\vartheta}$$

$$\vec{S} = S\,\hat{n}$$



$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{i}\cdot\hat{j} = \hat{k}\cdot\hat{j} = \hat{i}\cdot\hat{k} = 0$$

$$\hat{k} = \hat{i}\times\hat{j}, \quad \hat{i} = \hat{j}\times\hat{k}, \quad \hat{j} = \hat{k}\times\hat{i}$$

ορθοκανονική δεξιόστροφη βάση

$$\boxed{\vec{c} = \vec{a}\times\vec{b} = (\vec{a}\times\vec{b})_x\,\hat{i} + (\vec{a}\times\vec{b})_y\,\hat{j} + (\vec{a}\times\vec{b})_z\,\hat{k} = c_x\hat{i} + c_y\hat{j} + c_z\hat{k}}$$

$$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}, \quad \vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

$$m(\vec{a}\times\vec{b}) = (m\vec{a})\times\vec{b} = \vec{a}\times(m\vec{b})$$

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}, \quad m(\vec{a}\cdot\vec{b}) = (m\vec{a})\cdot\vec{b} = \vec{a}\cdot(m\vec{b})$$

$$\boxed{\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}$$

Πίνακας εξωτερικού γινομένου

ΔΧ

$$\vec{c} = \vec{a}\times\vec{b} = (a_x\hat{i}+a_y\hat{j}+a_z\hat{k})\times(b_x\hat{i}+b_y\hat{j}+b_z\hat{k}) =$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\hat{i} + (-1)\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\hat{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\hat{k}$$

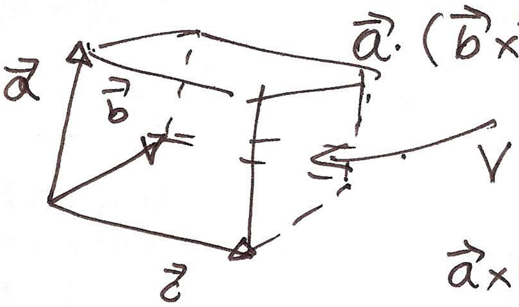
$$= \underline{(a_yb_z-a_zb_y)}\,\hat{i} + \underline{(a_zb_x-a_xb_z)}\,\hat{j} + \underline{(a_xb_y-a_yb_x)}\,\hat{k}$$

$$= c_x\,\hat{i} + c_y\,\hat{j} + c_z\,\hat{k}$$

βασικές ταυτότητες

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{c})$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \text{όγκος}$$

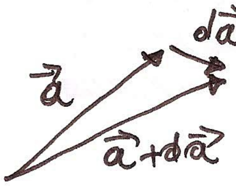


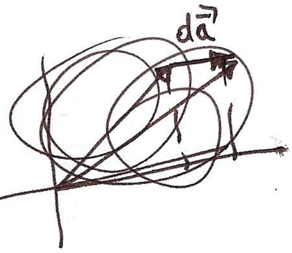
$$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

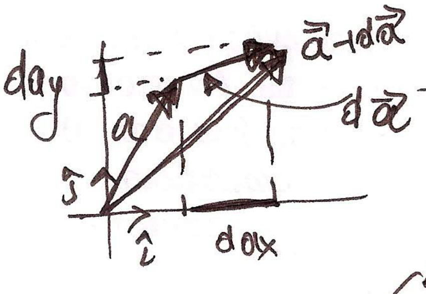
$$(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b}) = -\vec{a}(\vec{b}\cdot\vec{c}) + \vec{b}(\vec{a}\cdot\vec{c})$$

$$\boxed{d\vec{a} = da_x\,\hat{i} + da_y\,\hat{j} + da_z\,\hat{k}}$$

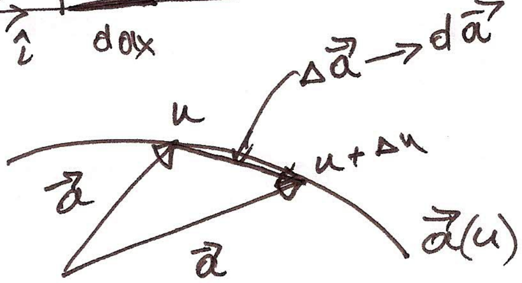




$$\boxed{\begin{gathered}\vec{a} = \vec{a}(u) \\ \frac{d\vec{a}}{du} = \frac{da_x}{du}\hat{i} + \frac{da_y}{du}\hat{j} + \frac{da_z}{du}\hat{k}\end{gathered}}$$



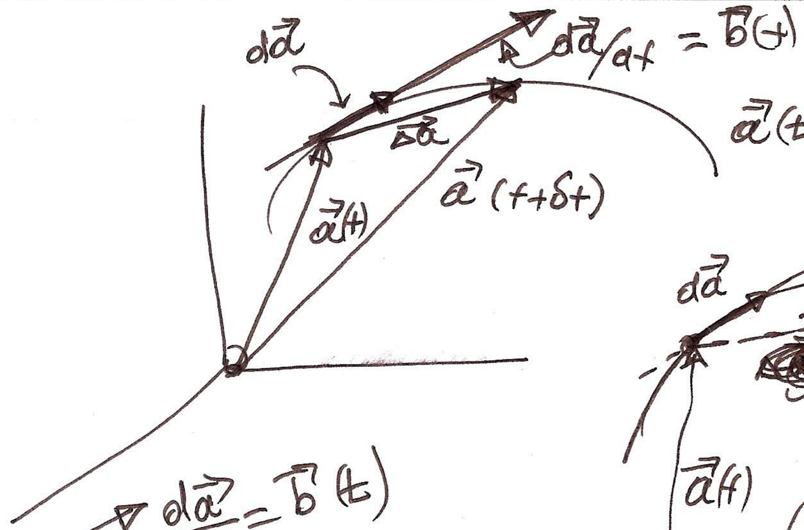
$$\boxed{\frac{d\vec{a}}{du} = \underset{\Delta u\to 0}{\text{όρ}}\ \frac{\vec{a}(u+\Delta u)-\vec{a}(u)}{\Delta u}}$$

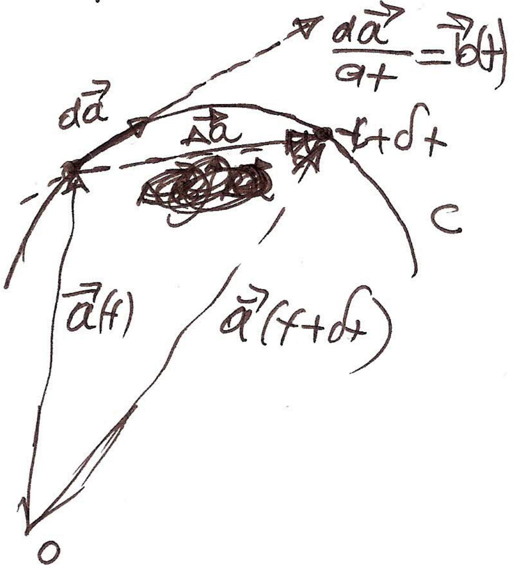


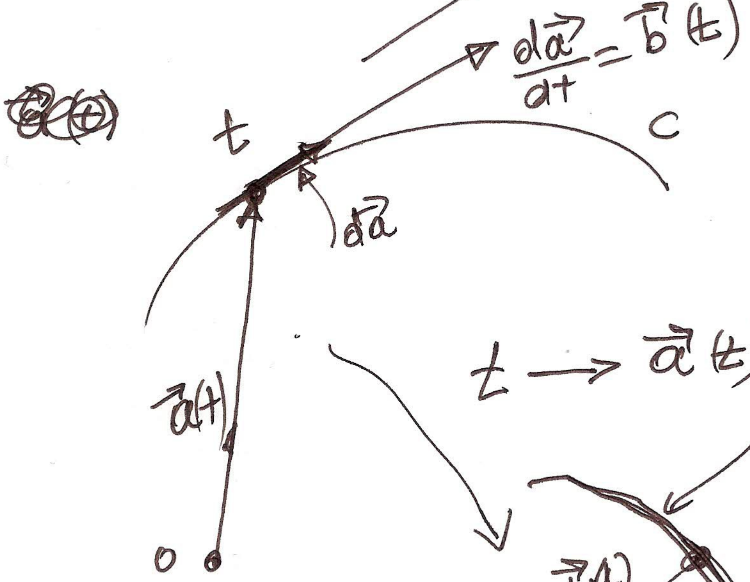
Δx

$$\vec{a} = \vec{a}(t)$$

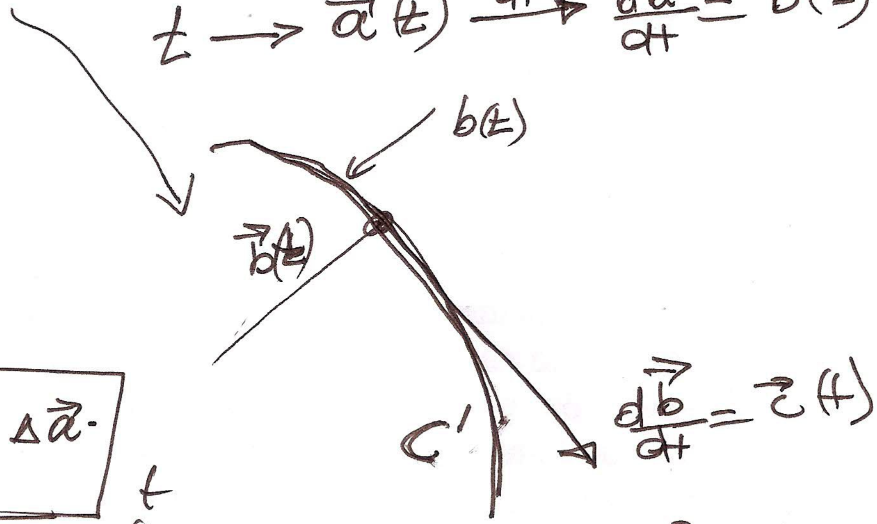
$$\boxed{\frac{d\vec{a}}{dt} = \vec{b}(t)}$$







$$t \longrightarrow \vec{a}(t) \xrightarrow{\frac{d}{dt}} \frac{d\vec{a}}{dt} = \vec{b}(t)$$



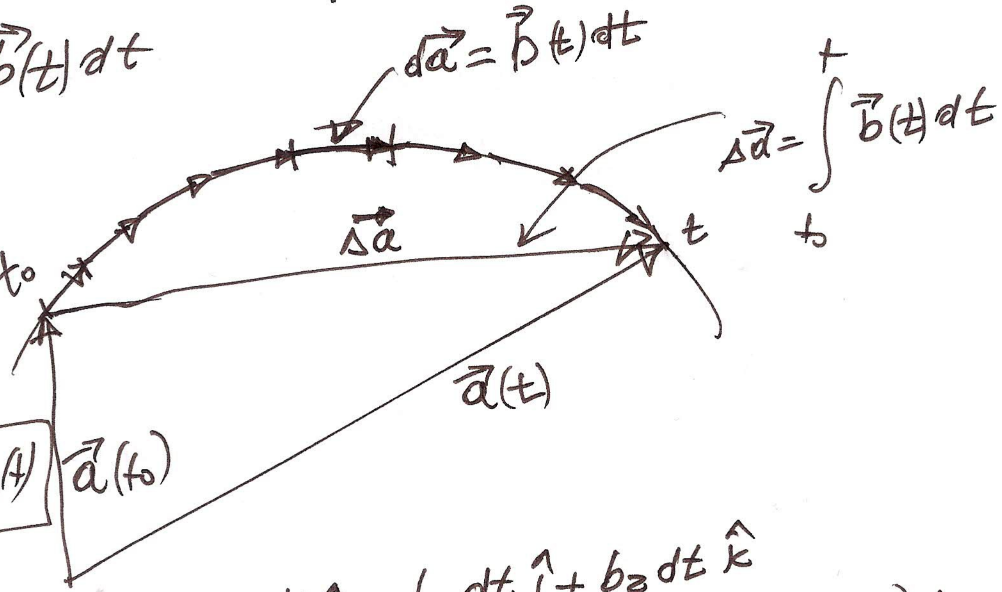
$$\boxed{\int \vec{b}(t)\,dt = \Delta\vec{a}}$$

$$\vec{a}(t) - \vec{a}(t_0) = \int_{t_0}^{t} \vec{b}(t)\,dt$$

$$\boxed{d\vec{a} = \vec{b}(t)\,dt}$$

$$da_x = b_x\,dt$$
$$da_y = b_y\,dt$$
$$da_z = b_z\,dt$$

$$\boxed{b_x = b_x(t),\ b_y = b_y(t),\ b_z = b_z(t)}$$



$$d\vec{a} = da_x\hat{i} + da_y\hat{j} + da_z\hat{k} = \underline{b_x\,dt}\,\hat{i} + \underline{b_y\,dt}\,\hat{j} + \underline{b_z\,dt}\,\hat{k}$$

$$\Delta\vec{a} = \int_{t_0}^{t} d\vec{a} = \int_{t_0}^{t}\left(da_x\hat{i} + da_y\hat{j} + da_z\hat{k}\right) = \left(\int b_x\,dt\right)\hat{i} + \left(\int b_y\,dt\right)\hat{j} + \left(\int b_z\,dt\right)\hat{k}$$

$$= \left(\int_{t_0}^{t} b_x(t)\,dt\right)\hat{i} + \left(\int_{t_0}^{t} b_y(t)\,dt\right)\hat{j} + \left(\int_{t_0}^{t} b_z(t)\,dt\right)\hat{k}$$

ΔX

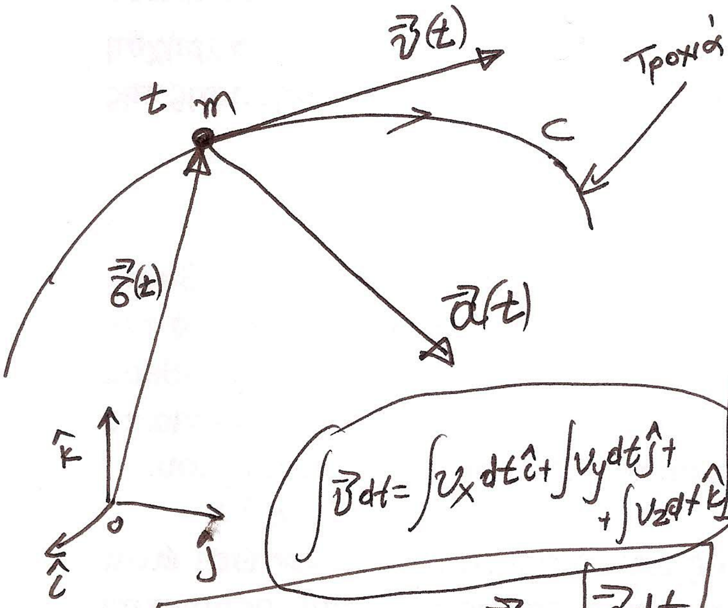
$d(\vec{a}+\vec{b}) = d\vec{a} + d\vec{b}$ $\qquad \dfrac{d(\vec{a}+\vec{b})}{du} = \dfrac{d\vec{a}}{du} + \dfrac{d\vec{b}}{du}$ $\left\{\begin{array}{l}\vec{a}=\vec{a}(u)\\ \vec{b}=\vec{b}(u)\end{array}\right\}$

$d(\lambda\vec{a}) = \lambda\, d\vec{a}$, ($\lambda$=σταθ) $\qquad \dfrac{d(\lambda\vec{a})}{du} = \lambda\dfrac{d\vec{a}}{du}$

$d(\phi\vec{a}) = (d\phi)\vec{a} + \phi(d\vec{a})$ $\qquad \dfrac{d(\phi\vec{a})}{du} = \dfrac{d\phi}{du}\vec{a} + \phi\dfrac{d\vec{a}}{du}$, $\left(\begin{array}{l}\phi=\phi(u)\\ \vec{a}=\vec{a}(u)\end{array}\right)$

$d(\vec{a}\cdot\vec{b}) = (d\vec{a})\cdot\vec{b} + \vec{a}\cdot(d\vec{b})$ $\qquad \dfrac{d(\vec{a}\cdot\vec{b})}{du} = \dfrac{d\vec{a}}{du}\cdot\vec{b} + \vec{a}\cdot\dfrac{d\vec{b}}{du}$, $\left(\begin{array}{l}\vec{a}=\vec{a}(u)\\ \vec{b}=\vec{b}(u)\end{array}\right)$

$d(\vec{a}\times\vec{b}) = \vec{a}\times d\vec{b} + d\vec{a}\times\vec{b}$ $\qquad \dfrac{d(\vec{a}\times\vec{b})}{du} = \dfrac{d\vec{a}}{du}\times\vec{b} + \vec{a}\times\dfrac{d\vec{b}}{du}$



$\dfrac{d\vec{r}}{dt} = \vec{v}(t)$, $\dfrac{d\vec{v}}{dt} = \vec{a}(t)$

$$\boxed{\begin{aligned} v_x(t) &= \frac{dr_x}{dt} = \frac{dx}{dt} \\ v_y(t) &= \frac{dr_y}{dt} = \frac{dy}{dt} \\ v_z &= \frac{dr_z}{dt} = \frac{dz}{dt} \end{aligned}}$$

$\int\vec{v}dt = \int v_x dt\,\hat{i} + \int v_y dt\,\hat{j} + \int v_z dt\,\hat{k}$

$a_x(t) = \dfrac{dv_x}{dt}$, $a_y(t) = \dfrac{dv_y}{dt}$, $a_z = \dfrac{dv_z}{dt}$

$\boxed{d\vec{r} = \vec{v}dt \Longrightarrow \Delta\vec{r} = \int\vec{v}dt}$

$dx = v_x dt \Rightarrow \int dx = \int v_x dt \Rightarrow \Delta x = \int v_x(t)dt$

$dy = v_y dt \Longrightarrow \int dy = \int v_y dt \Longrightarrow \Delta y = \int v_y(t)dt$

$dz = v_z dt \Longrightarrow \int dz = \int v_z dt \Longrightarrow \Delta z = \int v_z(t)dt$

$d\vec{v} = \vec{a}dt \Longrightarrow \Delta\vec{v} = \int\vec{a}dt = \int a_x dt\,\hat{i} + \int a_y dt\,\hat{j} + \int a_z dt\,\hat{k}$

$dv_x = a_x dt \Rightarrow \int dv_x = \int a_x dt \Rightarrow \Delta v_x = \int a_x(t)dt$

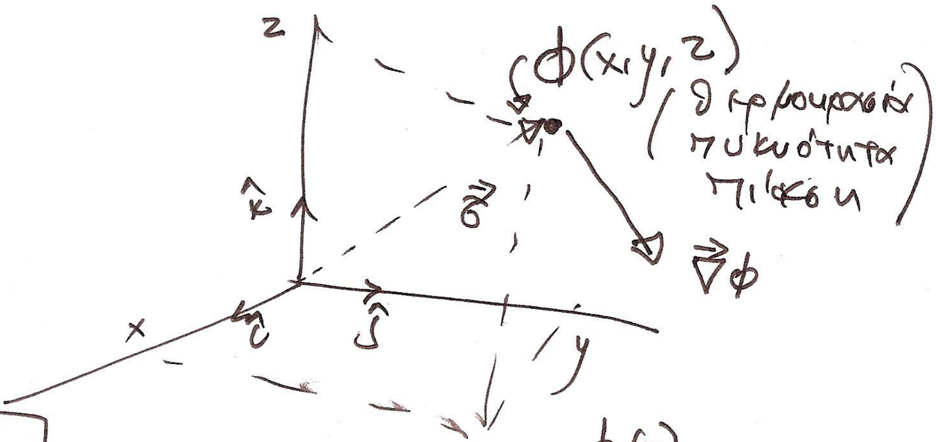
$dv_y = a_y dt \Rightarrow \int dv_y = \int a_y dt \Rightarrow \Delta v_y = \int a_y(t)dt$

$dv_z = a_z dt \Rightarrow \int dv_z = \int a_z dt \Rightarrow \Delta v_z = \int a_z(t)dt$

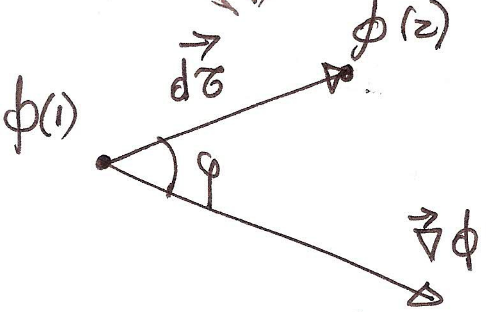
Συναρτήσεις πολλών μεταβλητών (Πεδία)

$$\phi = \phi(\vec{r}) = \phi(x\hat{i} + y\hat{j} + z\hat{k}) = \phi(x,y,z)$$

(βαθμωτό πεδίο)



$$\boxed{d\phi = \vec{\nabla}\phi \cdot d\vec{r}}$$



$\Delta\phi = \vec{\nabla}\phi \cdot \Delta\vec{r}$ (για $\Delta\vec{r} \to d\vec{r}$)

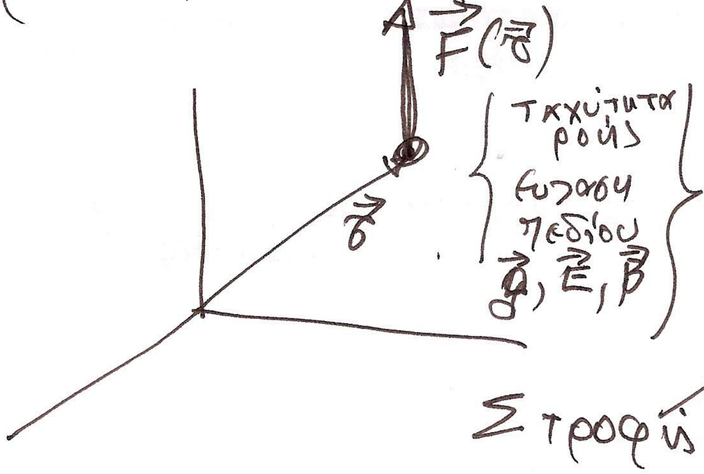
$$\Delta\phi = \phi(2) - \phi(1) \Longrightarrow d\phi$$

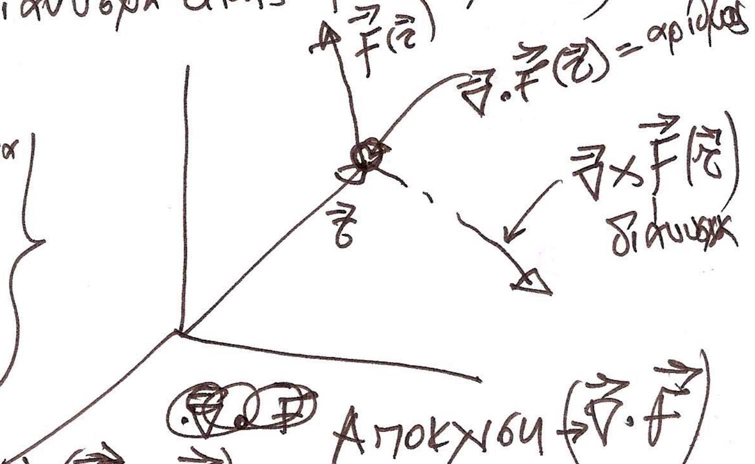
$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\phi(x,y,z) = x y^2 z^2 \Longrightarrow \vec{\nabla}\phi \; ;$$

$$\boxed{\begin{aligned}\vec{F}(\vec{r}) &= F_x(\vec{r})\hat{i} + F_y(\vec{r})\hat{j} + F_z(\vec{r})\hat{k} \\ &= F_x(x,y,z)\hat{i} + F_y(x,y,z)\hat{j} + F_z(x,y,z)\hat{k}\end{aligned}}$$

(Διανυσματική συνάρτηση διανυσματικής μεταβλητής)

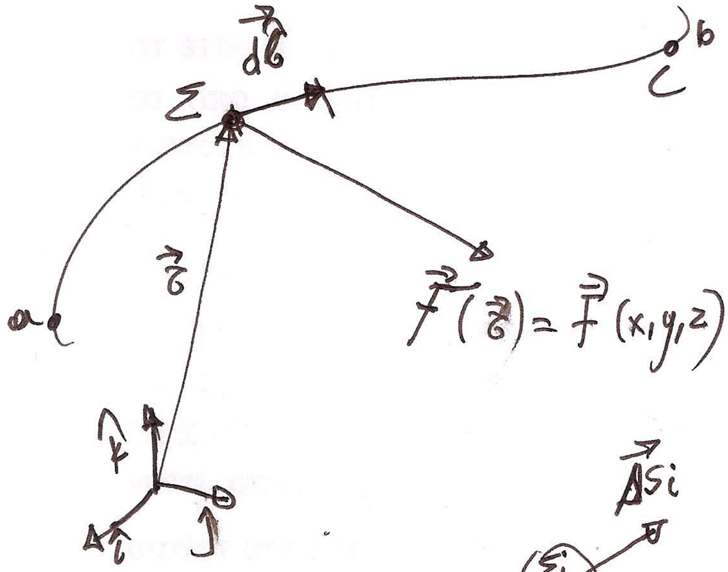




Απόκλιση $(\vec{\nabla}\cdot\vec{F})$

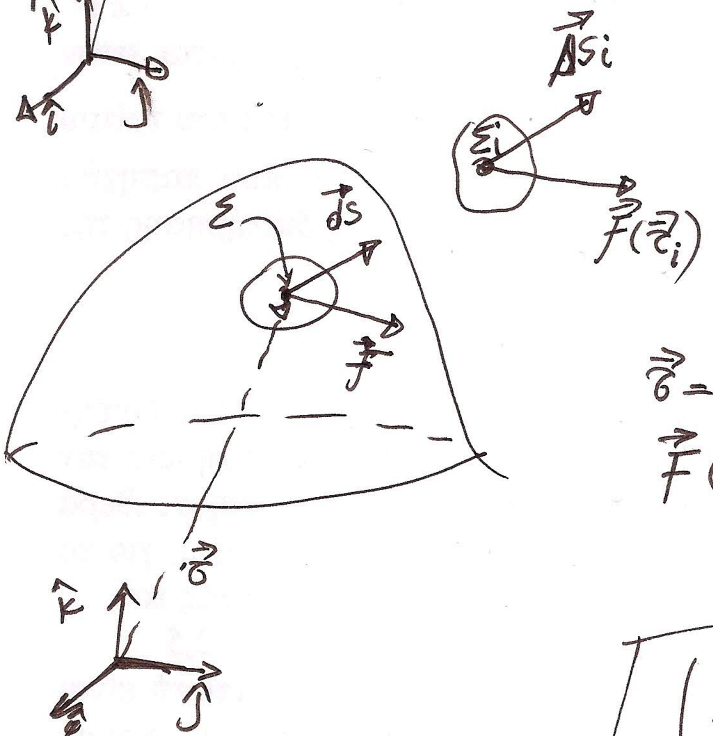
Στροφή $(\vec{\nabla}\times\vec{F})$

Επικαμπύλιο – επιφανειακό – κυβικό ολοκλήρωμα.



$$\boxed{\int_{c\,a}^{b} \vec{F}\cdot d\vec{\sigma} = \text{ορ}_{N\to\infty} \sum_{i=1}^{N} \vec{F}_i \cdot d\vec{\sigma}_i}$$

$$\vec{\sigma} = x\hat{i} + y\hat{j} + z\hat{k}$$

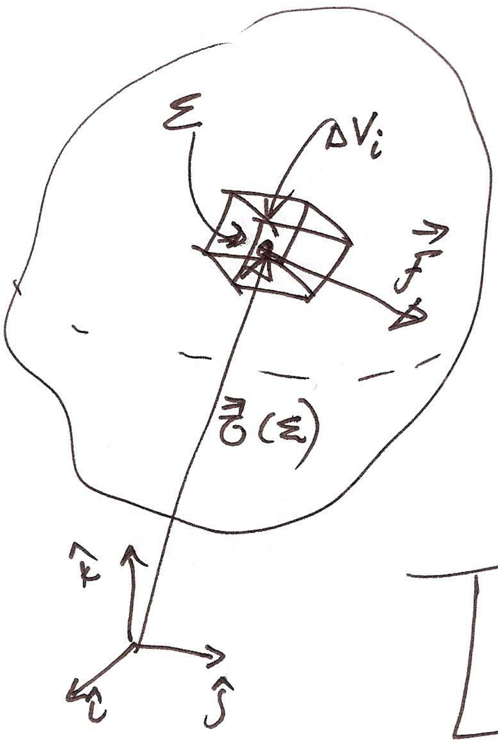


$$\boxed{\int_S \vec{F}\cdot d\vec{S} = \text{ορ}_{N\to\infty} \sum_{i=1}^{N} \vec{F}_i \cdot \Delta\vec{S}_i}$$

$$\vec{\sigma} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F}(\vec{\sigma}) = \vec{F}(x,y,z)$$

$$\boxed{\int_V \vec{F}\, dV = \text{ορ}_{N\to\infty} \sum_{i=1}^{N} \vec{F}_i \,\Delta V_i}$$



$$\cancel{\Delta V = \Delta x \Delta y \Delta z}$$

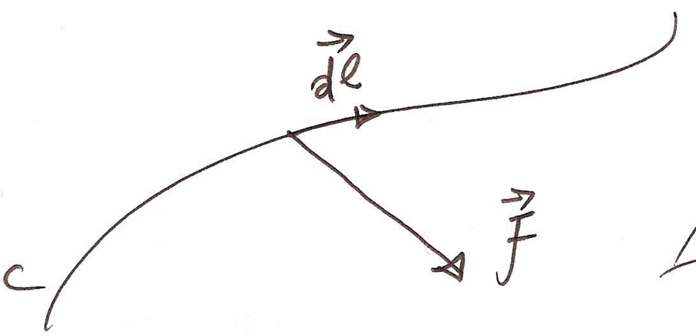
$$\cancel{\vec{\sigma}(\Sigma) = x\hat{i} + y}$$

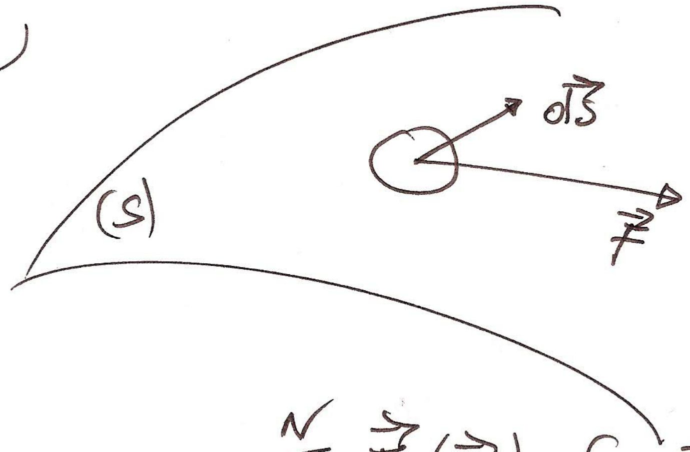
$$\vec{\sigma}(\Sigma) = x(\Sigma)\hat{i} + y(\Sigma)\hat{j} + z(\Sigma)\hat{k}$$
$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\boxed{\Sigma \longrightarrow \vec{\sigma}(\Sigma) = x(\Sigma)\hat{i} + y(\Sigma)\hat{j} + z(\Sigma)\hat{k}}$$

$\vec{\nabla}\cdot\vec{F}(\vec{r})$ απόκλιση

$\vec{\nabla}\times\vec{F}(\vec{r})$ στροφή



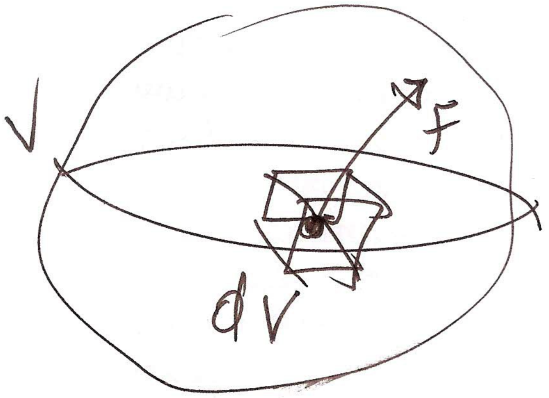


$$\lim_{N\to\infty}\sum_{i=1}^{N}\vec{F}_i\cdot\vec{\Delta l}_i=\int\vec{F}\cdot\vec{dl}$$

επικαμπύλιο ολοκλήρωμα

$$\lim_{N\to\infty}\sum_{i=1}^{N}\vec{F}_i\cdot(\vec{\Delta S}_i)=\int_S\vec{F}\cdot\vec{dS}$$

επιφανειακό ολοκλήρωμα



$$\int\rho\, dV=\sum\rho\,\Delta V$$

ρ = βαθμωτό πεδίο

$$\int\vec{F}\, dV=\sum\vec{F}\,\Delta V$$

F = διανυσματικό πεδίο

(κυβικό ολοκλήρωμα)

Gauss $$\int_V\vec{\nabla}\cdot\vec{F}\, dV=\oint_S\vec{F}\cdot\vec{dS}=\text{ροή } F=\Phi_F$$

Stokes $$\int_S\vec{\nabla}\times\vec{F}\cdot\vec{dS}=\oint_C\vec{F}\cdot\vec{dl}=\text{στροβιλισμός } \Sigma_F$$

$$\boxed{\begin{aligned}
&\bullet\ d(\phi\vec{A}) = \phi(d\vec{A}) + (d\phi)\vec{A} \Rightarrow \frac{d}{du}(\phi\vec{A}) = \frac{d\phi}{du}\vec{A} + \phi\frac{d\vec{A}}{du}\\
&\bullet\ d(\vec{A}\cdot\vec{B}) = d\vec{A}\cdot\vec{B} + \vec{A}\cdot d\vec{B} \Rightarrow \frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}\\
&\bullet\ d(\vec{A}\times\vec{B}) = d\vec{A}\times\vec{B} + \vec{A}\times d\vec{B} \Rightarrow \frac{d}{du}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}
\end{aligned}}$$

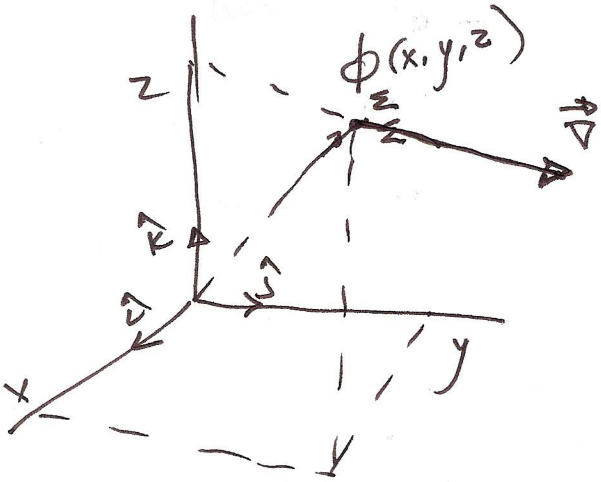
Συναρτήσεις πολλών μεταβλητών

$\phi(x,y,z)$

βαθμωτό πεδίο

$$\vec{A} = A_1(x,y,z)\hat{i} + A_2(x,y,z)\hat{j} + A_3(x,y,z)\hat{k}$$
$$= A_x(x,y,z)\hat{i} + A_y(x,y,z)\hat{j} + A_z(x,y,z)\hat{k}$$

Διανυσματικό πεδίο



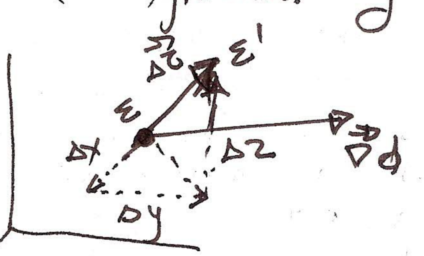
$$\phi = x^2yz^3$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3$$

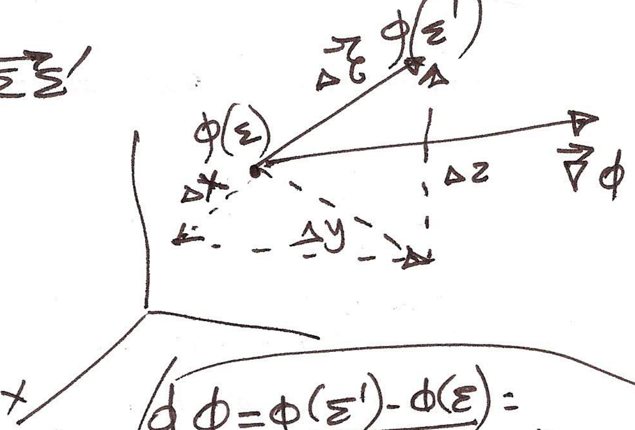
$$\frac{\partial\phi}{\partial y} = x^2z^3$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\frac{\partial\phi}{\partial x} = \left(\frac{\Delta\phi}{\Delta x}\right)_{y,z=\sigma\tau\alpha\theta.} \qquad \frac{\partial\phi}{\partial y} = \left(\frac{\Delta\phi}{\Delta y}\right)_{x,z=\sigma\tau\alpha\theta} \qquad , \frac{\partial\phi}{\partial z} = \left(\frac{\Delta\phi}{\Delta z}\right)_{x,y=\sigma\tau\alpha\theta}$$



$$\Delta\vec{r} = \vec{\Sigma\Sigma'}$$



$$\boxed{\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}}$$

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$ τελεστής βαθμίδας

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$d\phi = \phi(\Sigma') - \phi(\Sigma) = \vec{\nabla}\phi\cdot\Delta\vec{r} = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \frac{\partial}{\partial x}(x^2yz^3)\,dx + \frac{\partial}{\partial y}(x^2yz^3)\,dy + \frac{\partial}{\partial z}(x^2yz^3)\,dz$$

$$= (2xyz^3)\,dx + (x^2z^3)\,dy + (3x^2yz^2)\,dz$$

$$= \phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x,y,z) = d\phi$$

(για $\Delta x, \Delta y, \Delta z$ απειροστά)

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

$$\vec{A}(x,y,z) \Longrightarrow \begin{array}{l} \vec{\nabla}\cdot\vec{A} = \text{απόκλιση } \vec{A} \\ \vec{\nabla}\times\vec{A} = \text{στροφή } \vec{A} \end{array}$$

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(A_{\cancel{x}}\,\hat{i} + A_y\,\hat{j} + A_3\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(A_1) + \frac{\partial A_2}{\partial y} + \frac{\partial}{\partial z}(A_3) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\boxed{\vec{A} = xz\,\hat{i} - y^2\,\hat{j} + 2x^2y\,\hat{k}} \Longrightarrow \vec{\nabla}\cdot\vec{A} = \frac{\partial(xz)}{\partial x}\hat{i} + \frac{\partial(-y^2)}{\partial y}\hat{j} + \frac{\partial(2x^2y)}{\partial z}\hat{k}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial(xz)}{\partial x} + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(2x^2y)$$

$$= z - 2y + 0 = z - 2y = \text{αριθμός}$$

$$\vec{\nabla}\cdot\vec{A}(x,y,z) = \phi(x,y,z) = \text{βαθμωτό πεδίο}$$

$$\vec{\nabla}\times\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right) =$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{j}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

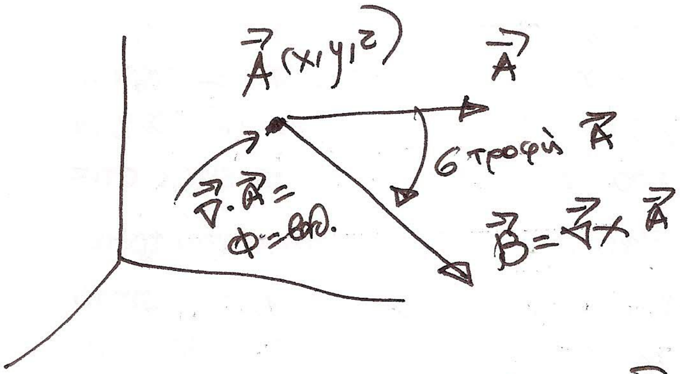
$$\vec{A} = xz\,\hat{i} - y^2\hat{j} + 2x^2y\,\hat{k}$$

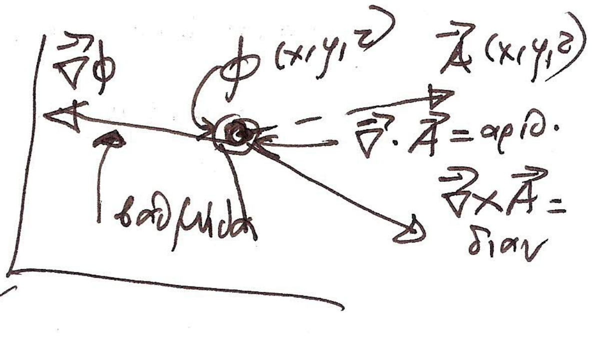
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y^2 & 2x^2y \end{vmatrix} = \dots\dots \quad 2x^2\hat{i} + (x-4xy)\hat{j} + 0\,\hat{k}$$

διανυσματικό πεδίο

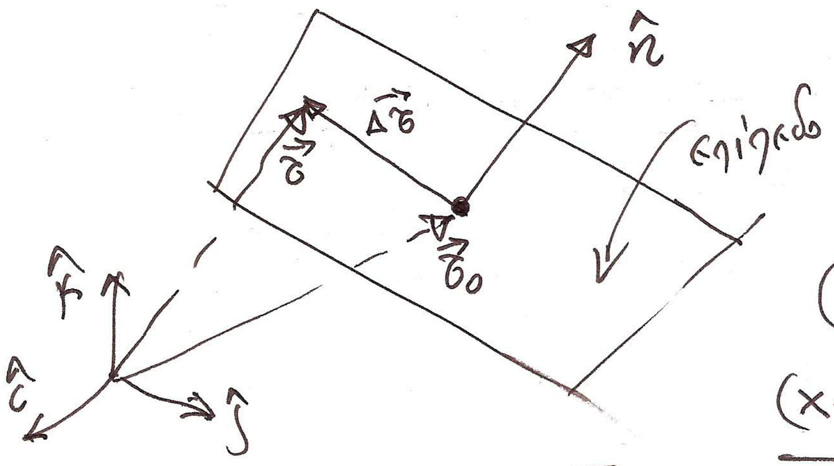
$$\vec{\nabla}\times\vec{A} = \vec{B}$$

$$\vec{A}(x,y,z) \xrightarrow{\vec{\nabla}\times} \vec{\nabla}\times\vec{A} = \vec{B}(x,y,z)$$





Επιφάνεια στό χώρο



$$\Delta\vec{r} = \vec{r} - \vec{r}_0$$

$$\Delta\vec{r}\cdot\hat{n} = 0$$

εξίσωση επιπέδου

$$(\vec{r}-\vec{r}_0)\cdot\hat{n} = 0$$

$$(x-x_0)\cdot n_x + (y-y_0)\cdot n_y + (z-z_0)n_z = 0$$

$$\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

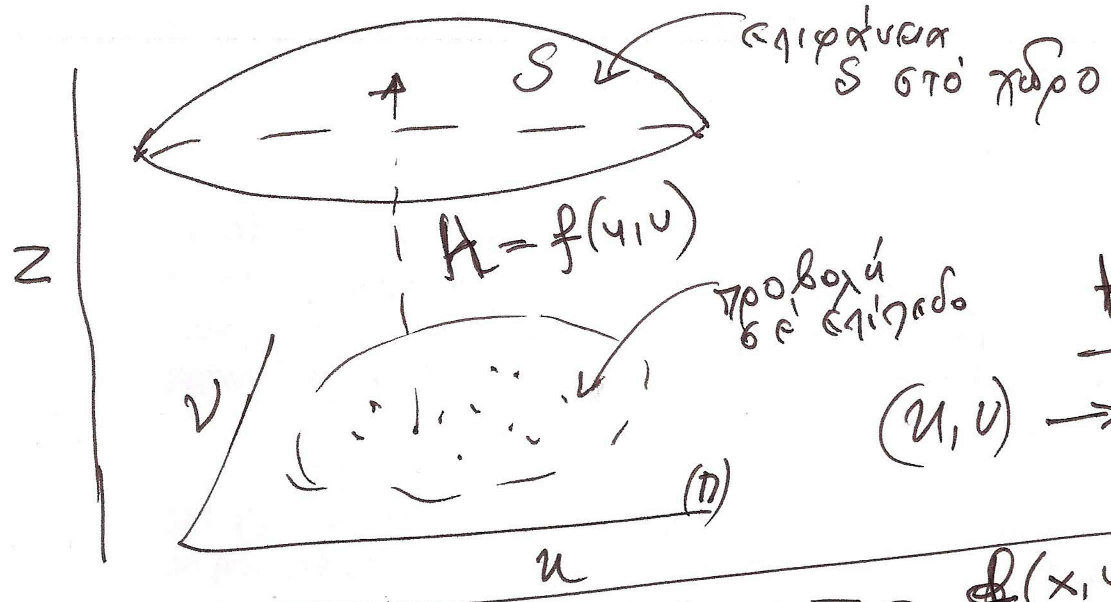
$$\boxed{\begin{aligned} \vec{a}\cdot\vec{b} &= a_xb_x + a_yb_y + a_zb_z \\ |\vec{a}|^2 &= a_x^2 + a_y^2 + a_z^2 \end{aligned}}$$

$$\boxed{(x-x_0)n_1 + (y-y_0)n_2 + (z-z_0)n_3 = 0 \quad \text{ή} \quad \phi(x,y,z) = 0}$$

εξίσωση επιπέδου

$$\phi(x,y,z) = 0 \quad \text{ή} \quad \phi(x,y,z) = \text{σταθ.}$$

(Εξίσωση επιφάνειας)

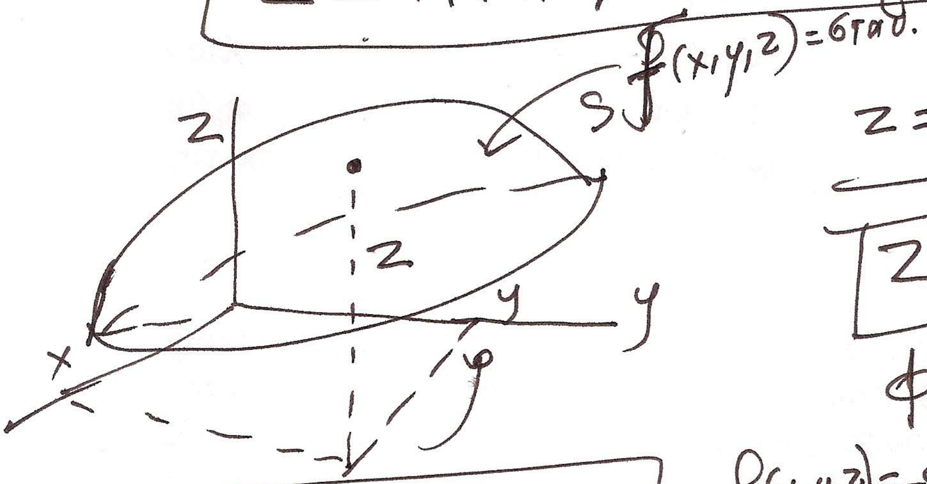


$$H = h(u,v) = z$$

$$(u,v) \rightarrow H(u,v) = z$$

$$\boxed{z = h(u,v) \quad \text{ή} \quad z = f(x,y)}$$

εξίσωση επιφάνειας



$$z = z(x,y) \quad \text{ή} \quad z = f(x,y)$$

$$\boxed{z - f(x,y) = 0}$$

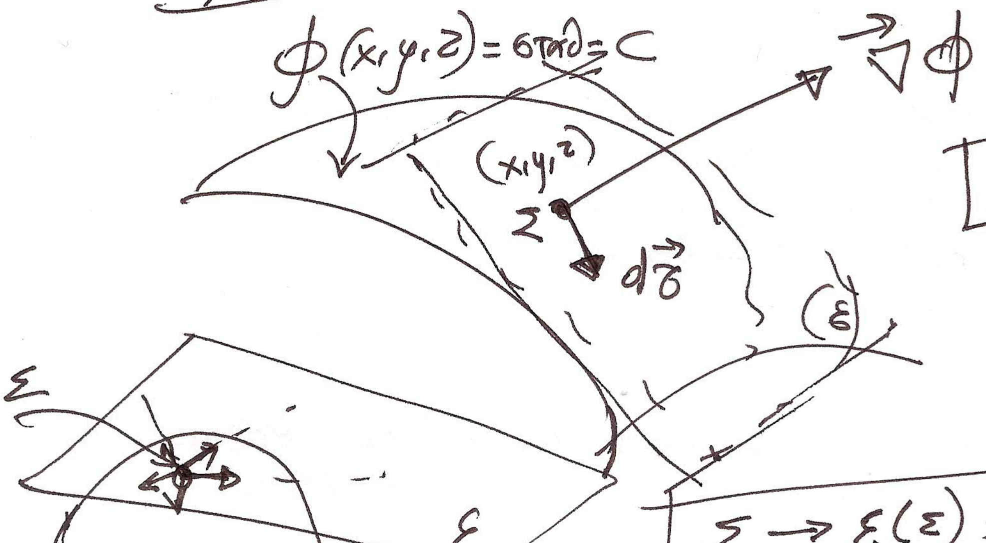
$$\phi(x,y,z) = C = \text{σταθ.}$$

$$f(x,y,z) = \phi(x,y,z) - C = 0$$

$$\boxed{F(x,y,z) = \text{σταθ}}$$

$$\boxed{\phi(x,y,z) = \text{σταθ} = C}$$

εξίσωση επιφάνειας



$$\phi(x,y,z) = C$$

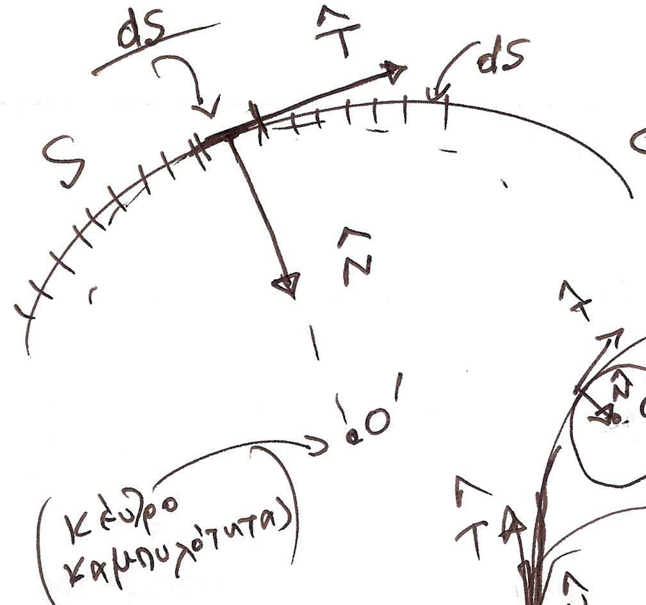
$$\boxed{d\phi = 0}$$

$$\vec{\nabla}\phi \cdot d\vec{r} = 0$$

$\vec{\nabla}\phi \perp \epsilon$ = εφαπτόμενο επίπεδο

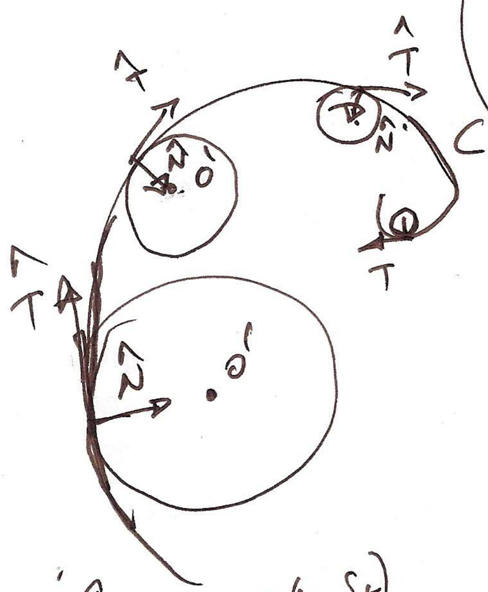
$\Sigma \rightarrow \varepsilon(\Sigma)$ = εφαπτόμενο επίπεδο στό Σ

$$\vec{\nabla}\phi \perp \varepsilon$$



$\hat{T}$ = μοναδιαίο εφαπτόμενο

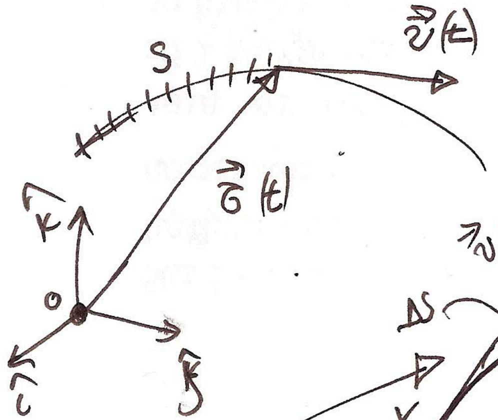
$\hat{N}$ = μοναδιαίο κεντρομόλο



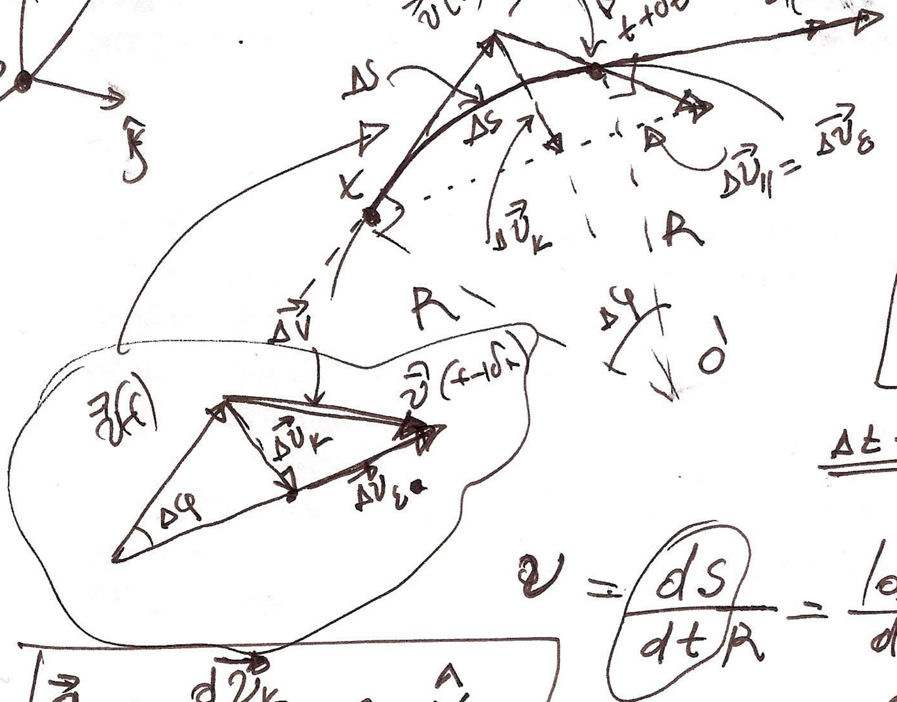
S = μήκος καμπύλης

S(t) = διανυόμενο μήκος

ds = στοιχειώδες μήκος



$$\frac{ds}{dt} = |\vec{v}| = \left|\frac{d\vec{r}}{dt}\right|$$



$$\Delta\vec{v} = \Delta\vec{v}_{\kappa} + \Delta\vec{v}_{\varepsilon}$$

$$\boxed{\frac{\Delta S}{R} = \frac{|\Delta\vec{v}_{\kappa}|}{|\vec{v}|}}$$

$\Delta t \to 0$

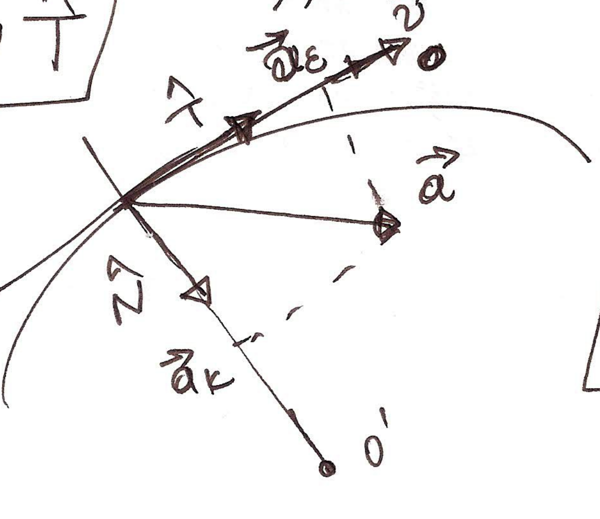
$$\boxed{\frac{dS}{R} = \frac{|d\vec{v}_{\kappa}|}{|\vec{v}|}}$$

$$v = \frac{ds}{dt}$$

$$\frac{ds}{dt\,R} = \frac{|d\vec{v}_{\kappa}|}{dt}\,\frac{1}{|\vec{v}|} = \frac{|\vec{a}_{\kappa}|}{|\vec{v}|} = \frac{a_{\kappa}}{v}$$

$$v/R = \frac{a_{\kappa}}{v} \Rightarrow \boxed{a_{\kappa} = \frac{v^2}{R}}$$

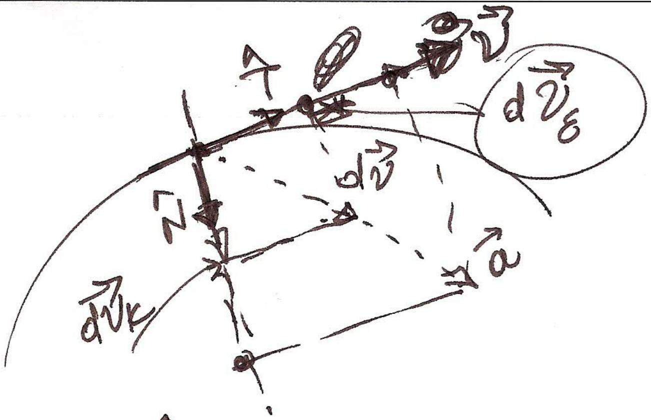
$$\boxed{\begin{aligned}\vec{a}_{\kappa} &= \frac{d\vec{v}_{\kappa}}{dt} = a_{\kappa}\hat{N} \\ \vec{v} &= \frac{ds}{dt}\hat{T} = v\hat{T}\end{aligned}}$$



$$\boxed{\frac{d\vec{v}_{\varepsilon}}{dt} = |\vec{a}_{\varepsilon}|\,\hat{T}}$$

$$\boxed{\begin{aligned}|\vec{a}_{\kappa}| &= \frac{v^2}{R} \\ |\vec{a}_{\varepsilon}| &= \frac{d\vec{v}_{\varepsilon}}{dt}\end{aligned}}$$

$$|\Delta\vec{v}_{\varepsilon}| = \Delta|\vec{v}|$$



$$\vec{v} = |\vec{v}|\,\hat{T} = \frac{ds}{dt}\,\hat{T}$$

$$\boxed{\frac{ds}{dt} = |\vec{v}|}$$

$$\boxed{\begin{aligned} d\vec{v}_\varepsilon &= |d\vec{v}_\varepsilon|\,\hat{T} = 0 \\ d\vec{v}_\kappa &= |d\vec{v}_\kappa|\,\hat{N} \end{aligned}}$$

$$|d\vec{v}_\varepsilon| = d|\vec{v}|$$

$$|d\vec{v}_\kappa| = \frac{v^2}{R}\,dt$$

$$\boxed{\left|\vec{a}_\varepsilon\right| = \left|\frac{d\vec{v}_\varepsilon}{dt}\right| = \frac{d|\vec{v}|}{dt} = \frac{d^2 s}{dt^2}}$$

$$\boxed{|\vec{a}_\kappa| = \frac{v^2}{R}}$$

$$d|\vec{v}| \neq |d\vec{v}|$$

$$\vec{v} = v\,\hat{T}$$

$$d\vec{v} = d\vec{v}_\varepsilon + d\vec{v}_\kappa$$

$$\boxed{d\vec{v}_\varepsilon = \vec{a}_\varepsilon\,dt\,,\quad d\vec{v}_\kappa = \vec{a}_\kappa\,dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{v}_\varepsilon}{dt} + \frac{d\vec{v}_\kappa}{dt}$$

$$= |\vec{a}_\varepsilon|\,\hat{T} + |\vec{a}_\kappa|\,\hat{N}$$

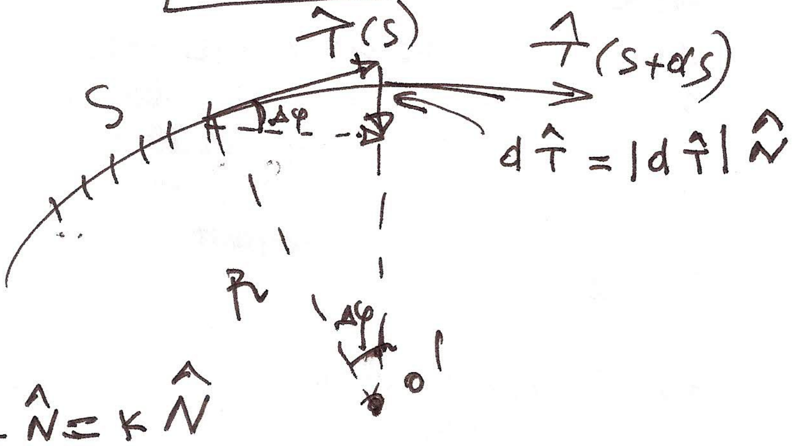
$$= \frac{d^2 s}{dt^2}\,\hat{T} + \frac{v^2}{R}\,\hat{N}$$

$$\boxed{\begin{aligned} &\frac{d|\vec{v}|}{dt} = \frac{|d\vec{v}_\varepsilon|}{dt} = \frac{d^2 s}{dt^2} \\ &|\vec{v}| = \frac{ds}{dt}\,,\quad \frac{d|\vec{v}|}{dt} = \frac{d^2 s}{dt^2} \end{aligned}}$$

K = καμπυλότητα

$$\boxed{K = \left|\frac{d\hat{T}}{ds}\right|} \qquad \boxed{K = \frac{1}{R}}$$

$$\boxed{\frac{d\hat{T}}{ds} = K\,\hat{N}}$$

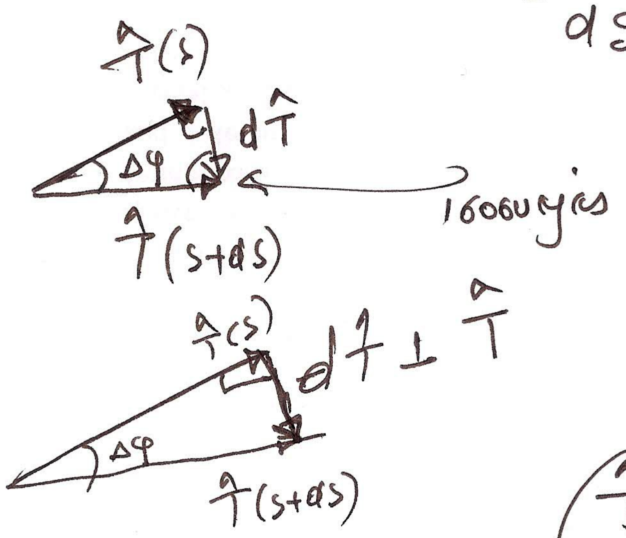


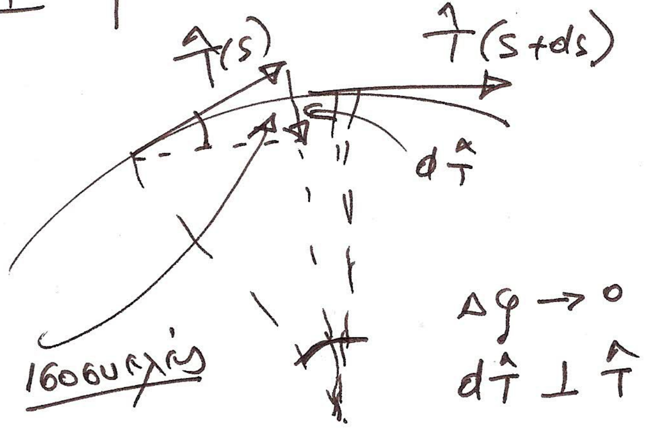
$$\frac{|d\hat{T}|}{ds} = \frac{|\hat{T}|}{R} = \frac{1}{R}$$

$$\frac{d\hat{T}}{ds} = \frac{|d\hat{T}|\,\hat{N}}{ds} = \frac{1}{R}\,\hat{N} = K\,\hat{N}$$

Άσκηση: ~~Να δ~~ Να δειχθεί ότι το

$$\frac{d\hat{T}}{ds} \perp \hat{T}$$





($\hat{T}$ = μοναδιαίο) $|\hat{T}| = |\hat{T}(s+ds)|$

$\hat{T}\cdot\hat{T} = 1$)

$$\frac{d(\hat{T}\cdot\hat{T})}{ds} = \frac{d(1)}{ds} = 0$$

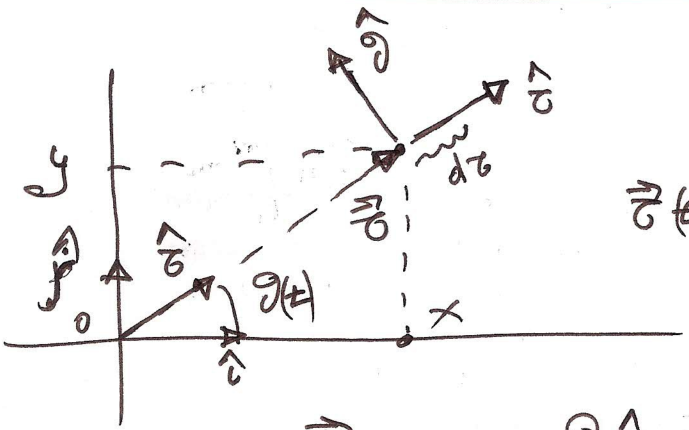
$$d(\hat{T}\cdot\hat{T}) = d\hat{T}\cdot\hat{T} + \hat{T}\cdot d\hat{T} = 2\,\hat{T}\cdot d\hat{T} = 0 \Longrightarrow$$

$$\boxed{\hat{T} \perp d\hat{T}}$$

Να δειχθούν οι σχέσεις:

$$\boxed{\begin{aligned}
&\bullet\ \vec{v} = |\vec{v}|\hat{T} = \frac{ds}{dt}\hat{T} = v\hat{T} \\
&\bullet\ \vec{a} = \frac{dv}{dT}\hat{T} + v\frac{d\hat{T}}{dt} = \frac{ds}{dt}\hat{T} + \frac{v^2}{R}\hat{N} \\
&\bullet\ \frac{d\hat{T}}{ds} = \frac{1}{R}\hat{N} = k\hat{N}
\end{aligned}}$$

Πολικές συνιστώσες



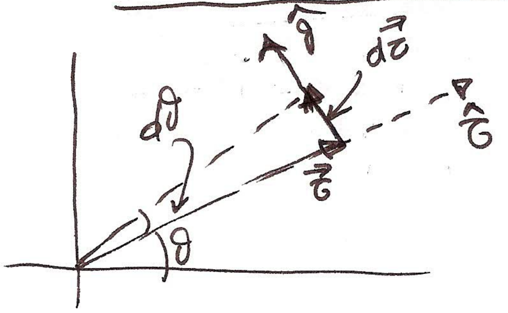
$\vec r(t), \theta(t)$

$$\hat\theta \perp \hat r$$

$$\vec r = |\vec r|\,\hat r = r\,\hat r$$

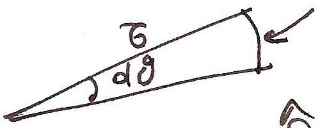
$$\vec r = r\cos\theta\,\hat i + r\sin\theta\,\hat j = x\,\hat i + y\,\hat j$$

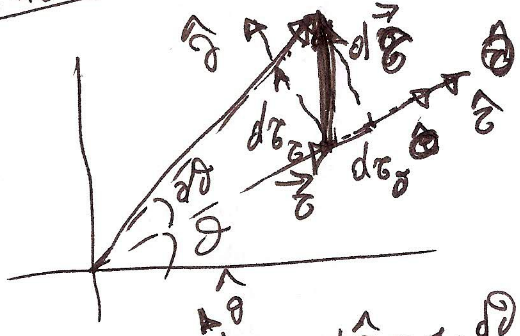
$$d\vec r = |d\vec r|\,\hat r = dr\,\hat r \quad , \theta = \text{σταθ.}$$



$$\boxed{d\vec r = (r\,d\theta)\,\hat\theta} \quad , \; r = \text{σταθ}$$

$$ds = r\,d\theta$$

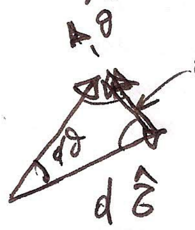




$$\boxed{d\vec r = dr\,\hat r + r\,d\theta\,\hat\theta}$$

$$d\vec r = dr_{(\theta)}\,\hat r + (r\,d\theta)_{(r)}\,\hat\theta$$

$$d\vec r = dr_\theta\,\hat r + dr_r\,\hat\theta =$$



$d\hat r = 1\cdot d\theta$

r = σταθ

$$d\hat r = (d\theta)\,\hat\theta = (1\cdot d\theta)\,\hat\theta$$

$$\boxed{\frac{d\hat r}{dt} = \frac{d\theta}{dt}\,\hat\theta = \dot\theta\,\hat\theta \quad , \; \dot\theta = \frac{d\theta}{dt}}$$

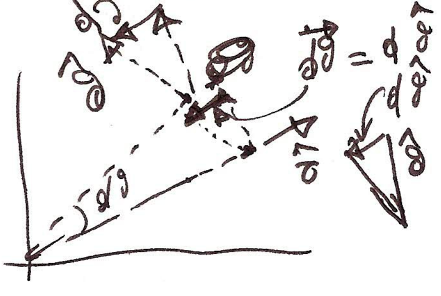
$$d(\hat r\cdot\hat r) = d(1) = 0$$

$$d\hat r \perp \hat r$$

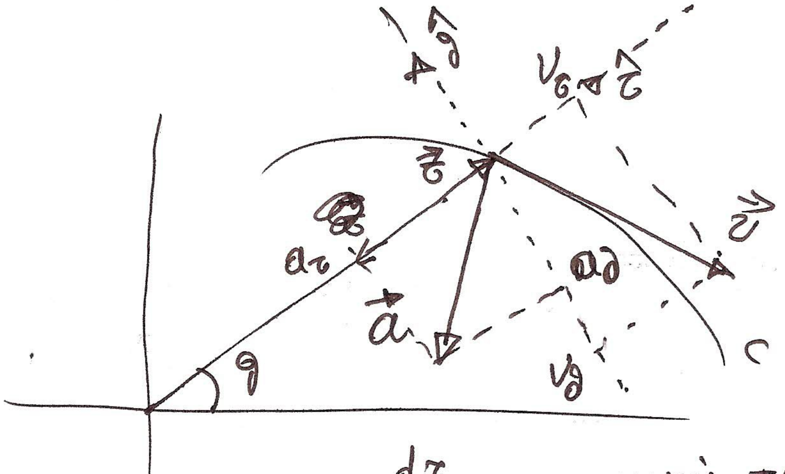
r = σταθ

$$d\hat\theta = (1\cdot d\theta)(-\hat r) = -d\theta\,\hat r$$

$$\boxed{\frac{d\hat\theta}{dt} = -\frac{d\theta}{dt}\,\hat r = -\dot\theta\,\hat r}$$



$d\hat\theta = d\theta$



$$\boxed{\begin{aligned}\vec{r} &= r\,\hat{r}\\ \vec{v} &= v_r\,\hat{r} + v_\theta\,\hat{\theta}\\ \vec{a} &= a_r\,\hat{r} + a_\theta\,\hat{\theta}\end{aligned}}$$

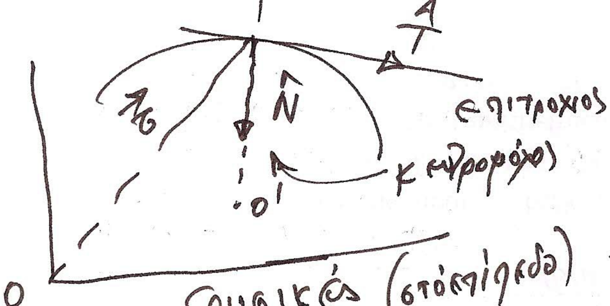
$\frac{d\theta}{dt} = \dot{\theta} =$ γωνιακή ταχύτητα

$v_r = \frac{dr}{dt} =$ ακτινική ταχύτητα

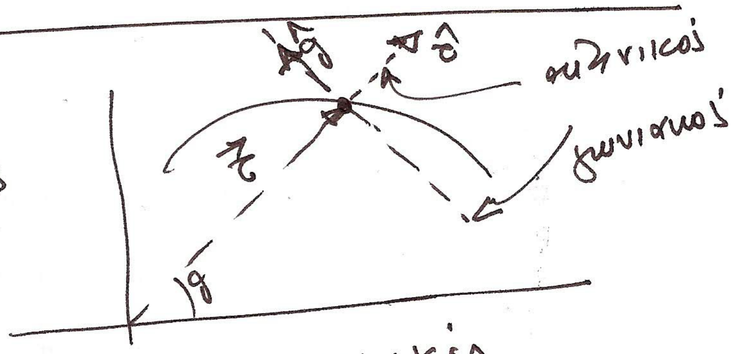
$v_\theta = r\frac{d\theta}{dt} = r\dot{\theta} =$ πολική ταχύτητα.

$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r\left(\dot{\theta}\right)^2 =$ ακτινική επιτάχυνση

$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\ddot{\theta} + 2\dot{r}\dot{\theta} =$ γωνιακή επιτάχυνση

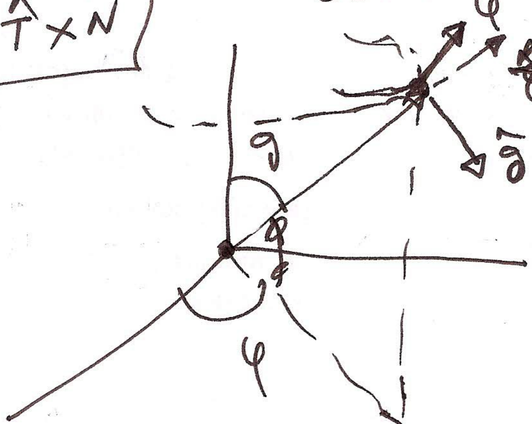


Φυσικές (στό επίπεδο) συνιστώσες



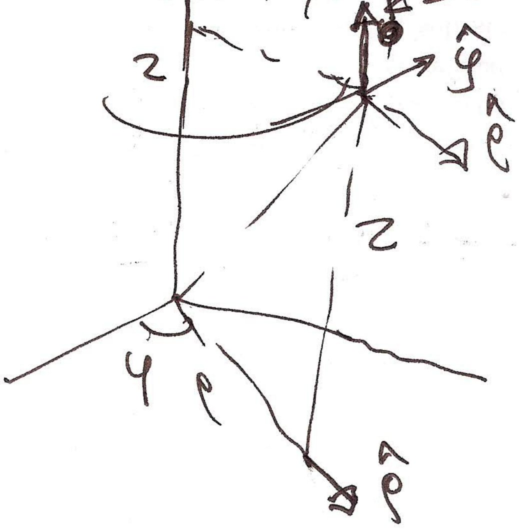
πολικές συνιστώσες

$$\boxed{\hat{B} = \hat{T}\times\hat{N}}$$



$(\hat{\varphi}, \hat{\theta}, \hat{r})$

φυσικές στό χώρο



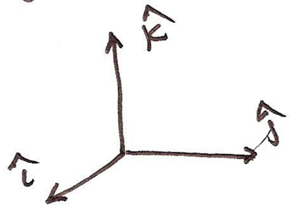
$(r, \varphi, \theta) \Rightarrow$ σφαιρικές

$(\rho, \varphi, z) \Rightarrow$ κυλινδρικές

Άσκηση ②

Αν $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $B = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ να δειχθεί ότι

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$



Απόδειξη

$$\vec{A}\times\vec{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\times(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) = A_1\hat{i}\times(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + A_2\hat{j}\times(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + A_3\hat{k}\times(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) =$$

$\hat{i}\times\hat{i} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{i}\times\hat{k} = -\hat{j}$

$\hat{j}\times\hat{j} = 0$, $\hat{j}\times\hat{i} = -\hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$

$\hat{k}\times\hat{k} = 0$, $\hat{k}\times\hat{i} = \hat{j}$, $\hat{k}\times\hat{j} = -\hat{i}$

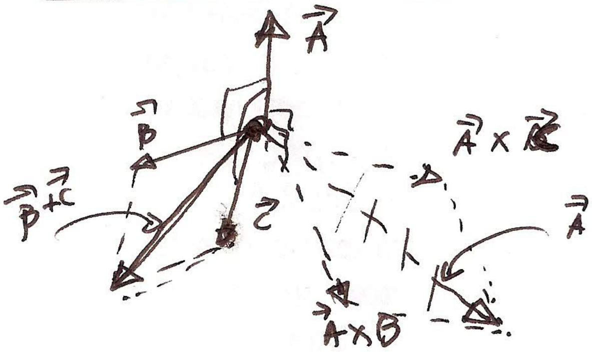
$$= (A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$$

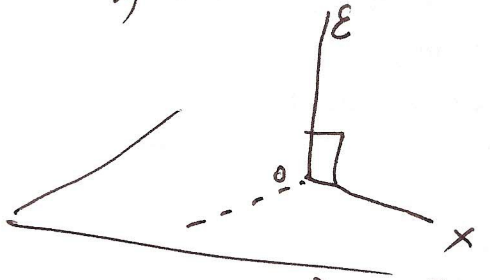
Άσκηση ①

$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ (να δειχθεί)

α) όταν $\vec{A}\perp\vec{B}$, $\vec{A}\perp\vec{C}$

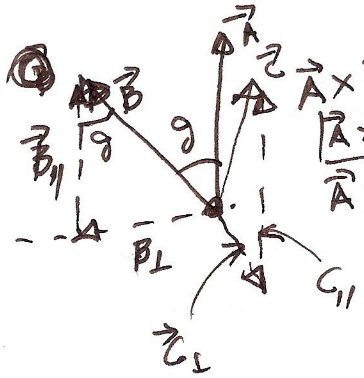
β) τυχαία





$$\vec{A}\times(\vec{B}_\perp + \vec{B}_\parallel) = \vec{A}\times\vec{B}_\perp$$

β)



$\vec{A}\times\vec{B} = AB\sin\theta = \vec{A}\times\vec{B}_\perp$

$\vec{A}\times\vec{B} = AB_\perp = |\vec{A}\times\vec{B}|$

$|\vec{A}\times\vec{B}| =$

$\vec{A}\times\vec{C} = \vec{A}\times\vec{C}_\perp$

$\vec{A}\times(\vec{B}_\perp + \vec{C}_\perp) = \vec{A}\times\vec{B}_\perp + \vec{A}\times\vec{C}_\perp$

$= \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

$(\vec{B}+\vec{C})_x = B_x + C_x$

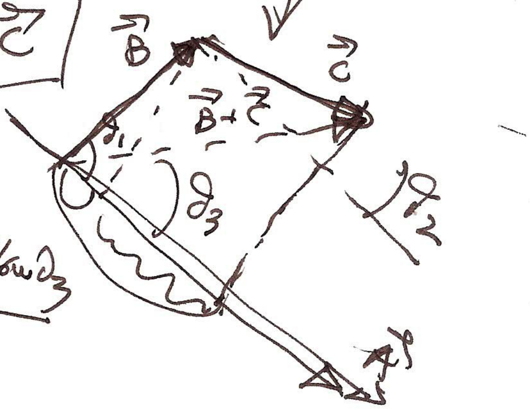
Άσκηση

$$\boxed{\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}}$$

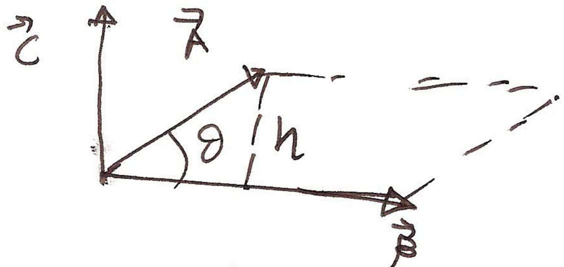
$\vec{A}\cdot\vec{B} = AB\cos\theta_1$

$\vec{A}\cdot\vec{C} = AC\cos\theta_2$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \underbrace{A|\vec{B}+\vec{C}|\cos\theta_3}_{(\vec{B}+\vec{C})_x}$$



Άσκηση ③ νά δειχθεί ότι τό εμβαδόν τού παραλληλογράμμου $(\vec{A},\vec{B})$ είναι $|\vec{A}\times\vec{B}|$

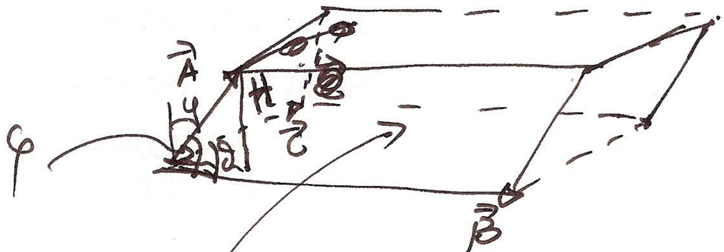


$$\vec{C} = \vec{A}\times\vec{B}$$

$$|\vec{C}| = |\vec{A}|\,|\vec{B}|\sin\theta = (|\vec{A}|\sin\theta)|\vec{B}| = h\,|\vec{B}| = S$$

$\left(S = B\,h = \text{βάση} \times \text{ύψος} \right)$

Άσκηση ④ $\vec{A}\cdot(\vec{B}\times\vec{C})$ = όγκος (V) παραλληλεπιπέδου $(\vec{A},\vec{B},\vec{C})$

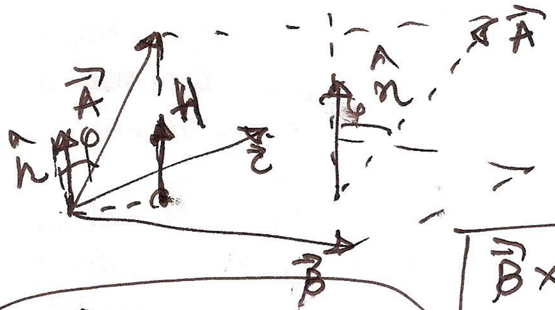


$$\vec{A}\cdot(\vec{B}\times\vec{C}) = |\vec{A}|\,|\vec{B}\times\vec{C}|\cos\varphi = V$$

$$\vec{B}\times\vec{C} = |\vec{B}\times\vec{C}|\,\hat{n}$$

$\hat{n}$ = μοναδιαίο $\perp (C,B)$

$$V = S(\vec{B},\vec{C})\cdot H = |\vec{B}\times\vec{C}|\cdot H = H\,(B\,C\sin\theta)\cos\varphi$$



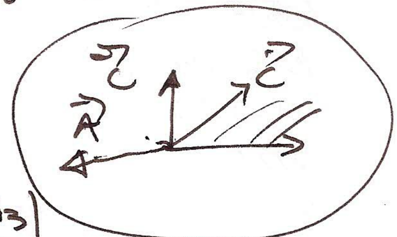
$$H = \vec{A}\cdot\hat{n}$$

$$V = H\,|\vec{B}\times\vec{C}| = (\vec{A}\cdot\hat{n})(|\vec{B}\times\vec{C}|) = \vec{A}\cdot(\hat{n}\,|\vec{B}\times\vec{C}|) = \vec{A}\cdot(\vec{B}\times\vec{C})$$

$$\boxed{\vec{B}\times\vec{C} = |\vec{B}\times\vec{C}|\,\hat{n}}$$

$\hat{n}$ = μοναδιαίο κάθετο στό επίπεδο $(\vec{B},\vec{C})$

$$\vec{C} = \vec{B}\times\vec{C} = \hat{n}\,|\vec{B}\times\vec{C}| = |\vec{C}|\,\hat{n}$$



Αν $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0 \Longrightarrow \vec{A},\vec{B},\vec{C}$ συνεπίπεδα

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} =$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

Ασκηση 5η

Νά υπολογιστή τό ολοκλήρωμα: $\int_{u=1}^{2} \vec{A}(u)\,du$, όπου

$$\vec{A}(u) = (3u^2-1)\hat{i} + (2u-3)\hat{j} + (6u^2-4u)\hat{k}$$

$$\int \vec{A}(u)\,du = \int \left(A_x du\,\hat{i} + A_y du\,\hat{j} + A_z du\,\hat{k}\right) = \int (3u^2-1)du\,\hat{i} + \int (2u-3)du\,\hat{j} + \int (6u^2-4u)\hat{k}$$

$$= (u^3-u)\hat{i} + (u^2-3u)\hat{j} + (2u^3-2u^2)\hat{k}\Big|_{u=1}^{2}$$

$$= (2^3-2)\hat{i} + (2^2-3\cdot 2)\hat{j} + (2\cdot 2^3 - 2\cdot 2^2)\hat{k} - \left\{(1^3-1)\hat{i} + (1^2-3\cdot 1)\hat{j} + (2\cdot 1^3 - 2\cdot 1^2)\hat{k}\right\}$$

$$= 6\hat{i} + 8\hat{k}$$

Ασκηση 6η

$\vec{A}(u),\ \vec{B}(u)$

Νά δειχθεί ότι: $\frac{d}{du}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$

$$d(\vec{A}\cdot\vec{B}) = \vec{A}(u+du)\cdot\vec{B}(u+du) - \vec{A}(u)\cdot\vec{B}(u) =$$

$$= \vec{A}(u+du)\cdot\vec{B}(u+du) - \vec{A}(u)\cdot\vec{B}(u+du) +$$

$$+ \vec{A}(u)\cdot\vec{B}(u+du) - \vec{A}(u)\cdot\vec{B}(u) =$$

$$= [\vec{A}(u+du) - \vec{A}(u)]\cdot(\vec{B}(u+du) +$$

$$+ \vec{A}(u)\cdot[\vec{B}(u+du) - \vec{B}(u)] = \quad (du \to 0$$

$$= d\vec{A}\cdot\vec{B}(u) + \vec{A}\cdot d\vec{B}$$

$$d(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}\cdot\vec{B} + \vec{A}\cdot d\vec{B}}{du} = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$$

$$\boxed{\begin{aligned}&\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}\\ &\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})\end{aligned}}$$

$$\vec{A}(u) \Longrightarrow \boxed{\frac{d\vec{A}}{du} = \frac{dA_x}{du}\hat{i} + \frac{dA_y}{du}\hat{j} + \frac{dA_z}{du}\hat{k}} \quad 1^{\eta}\ \pi\alpha\rho.$$

$$\boxed{\frac{d^2\vec{A}}{du^2} = \frac{d^2A_x}{du^2}\hat{i} + \frac{d^2A_y}{du}\hat{j} + \frac{d^2A_z}{du}\hat{k}} \quad 2^{\eta}\ \pi\alpha\rho.$$

$$\int \vec{A}(u)\,du = \int A_x(u)\,du\,\hat{i} + \int A_y(u)\,du\,\hat{j} + \int A_z(u)\,du\,\hat{k}$$

$\vec{A}(u) = \dfrac{d\vec{B}}{du}$ (αν υπάρχει $\vec{B}(u)$: $\dfrac{d\vec{B}}{du} = \vec{A}(u)$

$$\int_{u_1}^{u_2} \vec{A}(u)\,du = \int_{u_1}^{u_2} \frac{d\vec{B}}{du}\,du = \int d\vec{B} = \Delta\vec{B} = \vec{B}(u_2) - \vec{B}(u_1)$$

$$\vec{r}(t) \xrightarrow{\frac{d}{dt}} \vec{v}(t) = \frac{d\vec{r}}{dt} \Longrightarrow \vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 3e^{-2t}\hat{i} + 4\sin 3t\,\hat{j} + 5\cos 3t\,\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -6e^{-2t}\hat{i} + 12\cos 3t\,\hat{j} - 15\sin 3t\,\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 12e^{-2t}\hat{i} - 36\sin 3t\,\hat{j} - 45\cos 3t\,\hat{k}$$

$$\boxed{\int \vec{a}\,dt = \vec{v}(t) + \vec{C}_1 = \Delta\vec{v}, \quad \int \vec{v}(t)\,dt = \vec{r} + \vec{C}_2 = \Delta\vec{r}}$$

$$\Delta\vec{v} = \vec{v}(t) - \vec{v}(t_0) \qquad \Delta\vec{r} = \vec{r}(t) - \vec{r}(t_0)$$

Δίδεται

$$\vec{A} = (3x^2 - 6yz)\hat{i} + (2y + 3xz)\hat{j} + (1 - 4xyz^2)\hat{k}$$

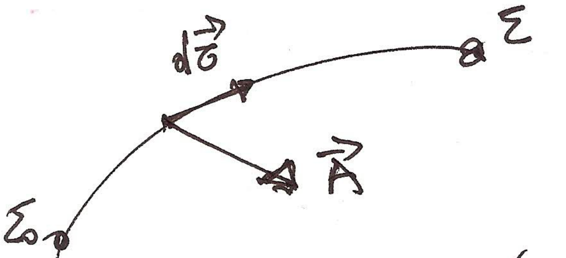
Να υπολογιστεί το ολοκλήρωμα:

$$\int_C \vec{A}\cdot d\vec{\sigma} \qquad \Sigma_0 = (0,0,0) \to \Sigma = (1,1,1)$$

$C: x = t,\ y = t^2,\ z = t^3, \qquad \vec{\sigma}(t) = t\,\hat{i} + t^2\hat{j} + t^3\hat{k}$

$$\cancel{\int \vec{A}\cdot d\vec{\sigma} = \int (3t^2 - 6t^5)\hat{i}\cdot dt\,\hat{i} + \int 2t^2 +}$$

$$\vec{A}\cdot d\vec{\sigma} = A_x\,dx + A_y\,dy + A_z\,dz = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$



$$d\vec{\sigma} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} = d(t)\,\hat{i} + d(t^2)\,\hat{j} + d(t^3)\,\hat{k}$$

$$= \hat{i}\,dt + \hat{j}(2t) + \hat{k}(3t^2)\,dt$$

$$\vec{A}\cdot d\vec{\sigma} = (3x^2 - 6yz)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz$$

$$+ (2t^2 + 3t\cdot t^3)dt + (1 - 4t\cdot t^2\cdot (t^3)^2)\cdot(3t^2)dt$$

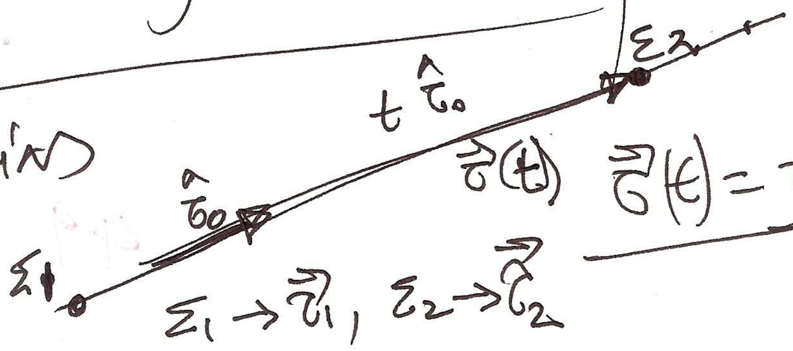
$$= (3t^2 - 6t^5)dt + 2t(2t^2 + 3t\cdot t^3)dt + (3t^2 - 12t^{11})dt$$

$$= (3t^2 - 6t^5)dt + (4t^3 + 6t^5)dt + (3t^2 - 12t^{11})dt$$

$$\int \vec{A}\cdot d\vec{\sigma} = \int_{t=0}^{t=1}(3t^2 - 6t^5)dt + \int_{t=0}^{t=1}(4t^3 + 6t^5)dt + \int_{t=0}^{t=1}(3t^2 - 12t^{11})dt = 2$$

$$\boxed{d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz}$$

Εξίσωση ευθείας



$$\vec{\sigma}(t) = t\,\hat{\tau}_0$$

$\Sigma_1 \to \vec{\sigma}_1,\ \Sigma_2 \to \vec{\sigma}_2$

$$\vec{A} = (2xy+z^3)\hat{i} + (x^2+2y)\hat{j} + (3xz^2-2)\hat{k}$$

Να δειχθεί ότι: $\vec{\nabla}\times\vec{A} = 0$, $\vec{A} = \vec{\nabla}\phi$, $\phi = ;$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2+2y & 3xz^2-2 \end{vmatrix} = 0$$

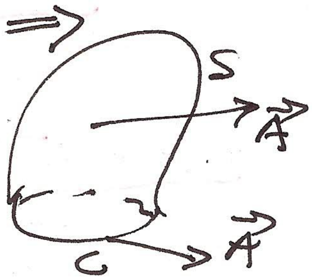
$$\left[\frac{\partial}{\partial y}(3xz^2-2) - \frac{\partial}{\partial z}(x^2+2y)\right]\hat{i} + \left[\frac{\partial}{\partial z}(2xy+z^3) - \frac{\partial}{\partial x}(3xz^2-2)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}(x^2+2y) - \frac{\partial}{\partial y}(2xy+z^3)\right]\hat{k} = (0-0)\hat{i} + (\overset{0}{\cancel{0}} + \cancel{3z^2} - \cancel{(3z^2)})\hat{j}$$

$$+ (\cancel{2x - 2x})\hat{k} = 0$$

$$\vec{\nabla}\times\vec{A} = 0 \Longrightarrow \oint_c \vec{A}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{S} = 0 \Longrightarrow$$

(stokes)



$$\oint_c \vec{A}\cdot d\vec{r} = 0 \Longrightarrow \boxed{\vec{A}\cdot d\vec{r} = d\phi}$$ (τέλειο διαφορικό)

$$\vec{A}\cdot d\vec{r} = A_x dx + A_y dy + A_z dz = (2xy+z^3)dx + (x^2+2y)dy + (3xz^2-2)dz$$

$$= d(x^2y + xz^3) + x^2dy + 2y\,dy + 3xz^2dz - 2dz$$

$$d(x^2y + xz^3) + d(y^2) + d(-2z) + x^2dy + 3xz^2dz$$

$$d(x^2y + xz^3) + d(y^2) + d(-2z)$$

$$d(x^2y + xz^3 + y^2 - 2z)$$

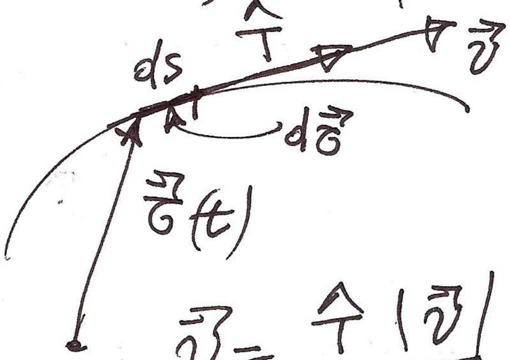
$$= d(\underbrace{x^2y + xz^3 + y^2 - 2z}_{\phi})$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

Άσκηση

$\vec{r} = 3\cos 2t\,\hat{i} + 3\sin 2t\,\hat{j} + (8t-4)\hat{k}$, Να βρεθεί

Το εφαπτόμενο $\hat{T}$



$$\boxed{\hat{T} = \frac{d\vec{r}}{ds}}$$

$$|d\vec{r}| = ds$$

$$\vec{v} = \hat{T}\,|\vec{v}|$$

$$\boxed{\frac{d\vec{r}}{ds}\cdot\cancel{\frac{ds}{dt} = \frac{d\vec{r}}{dt}\,|\vec{v}| = \hat{T}\cdot|\vec{v}|}}$$

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -6\sin 2t\,\hat{i} + 6\cos 2t\,\hat{j} + 8\hat{k}$$

$$|\vec{v}| = \frac{ds}{dt} = \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2 + 8^2} = \sqrt{100} = 10$$

$$\hat{T} = \frac{-6\sin 2t\,\hat{i} + 6\cos 2t\,\hat{j} + 8\hat{k}}{10} = -\frac{3}{5}\sin 2t\,\hat{i} + \frac{3}{5}\cos 2t\,\hat{j} + \frac{4}{5}\hat{k}$$

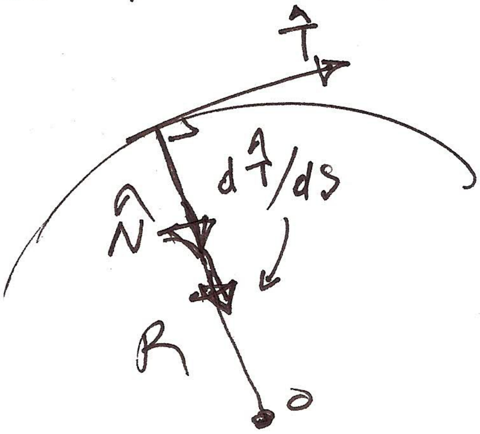
$$\vec{v} = |\vec{v}|\,\hat{T} = 10\left(\frac{\qquad}{10}\right) = \vec{v}\ldots$$

Να δειχθεί ότι $d\hat{T}/ds \perp \hat{T}$

$$\hat{T}\cdot\hat{T} = 1 \Rightarrow d(\hat{T}\cdot\hat{T}) = 0 \Rightarrow \hat{T}\cdot d\hat{T} + \hat{T}\cdot d\hat{T} = 0$$

$$\Rightarrow 2\hat{T}\cdot d\hat{T} = 0 \Rightarrow \hat{T}\cdot d\hat{T} = 0 \Rightarrow \hat{T} \perp d\hat{T}$$

$$\frac{d\hat{T}}{ds} \parallel d\hat{T} \Rightarrow \frac{d\hat{T}}{ds} \perp \hat{T}$$



$$\hat{N} = K\frac{d\hat{T}}{ds} \Rightarrow K = \frac{1}{R}$$

Να δειχθή ότι $\dfrac{d^2\vec{r}}{dt^2} = \dfrac{d|\vec{v}|}{dt}\hat{T} + \dfrac{v^2}{R}\hat{N}$

Δίδονται τα σημεία $\vec{r}_1, \vec{r}_2, \vec{r}_3$

Να βρεθή η εξίσωση του επιπέδου

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$
$$\vec{r}_3 = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$$

$$(\vec{r} - \vec{r}_2) \perp \vec{N}$$

$$(\vec{r} - \vec{r}_2)\cdot\left[(\vec{r}_1 - \vec{r}_2)\times(\vec{r}_3 - \vec{r}_2)\right] = 0$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\boxed{\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ x_3-x_2 & y_3-y_2 & z_3-z_2 \end{vmatrix} = 0}$$

εξίσωση επιπ.

$P_1 = (-1, 2, 4)$, $P_2 = (3, 1, -2)$ $P_3 = (2, -1, 1)$

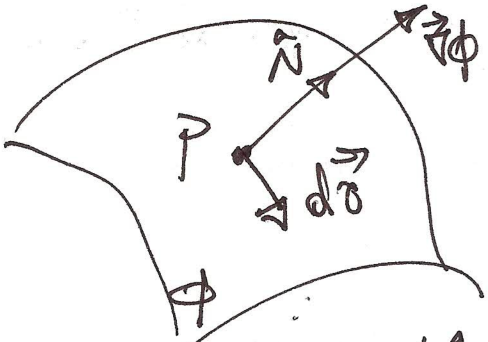
$$\boxed{5x + 2y + 3z = 11}$$

Να βρεθή η κάθετος στό επίπεδο:

$$\underline{2x^2+4yz-5z^2=10}$$ στό σημείο

$P=(3,-1,2)$ καί μοναδιαίο κάθετο

$\phi=$ σταθ $\quad \phi = 2x^2+4yz-5z^2$



$$d\phi = \vec\nabla\phi\cdot d\vec\sigma = 0 \Rightarrow \vec\nabla\phi \perp d\vec\sigma$$

$$\vec\nabla\phi = \frac{\partial\phi}{\partial x}\hat i + \frac{\partial\phi}{\partial y}\hat j + \frac{\partial\phi}{\partial z}\hat k =$$

$$\cancel{2x\hat i + 4z\hat j + 5z}$$

$$4x\hat i + 4z\hat j + (4y-10z)\hat k =$$

$$= 4\cdot3\,\hat i + 4\cdot2\,\hat j + (4\cdot(-1) - 10\cdot2)\hat k$$

$$= 12\hat i + 8\hat j - 24\hat k$$

$$\underline{\vec\nabla\phi = |\vec\nabla\phi|\,\hat N}$$

$$|\vec\nabla\phi| = \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]^{1/2}$$

$$\boxed{|\vec\nabla\phi| = \sqrt{12^2+8^2+(-24)^2} = 7}$$

$$\hat N = \frac{3\hat i + 2\hat j - 6\hat k}{7}$$

$$\boxed{\vec\nabla\cdot\vec A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$\nabla^2\phi =$;

$$\boxed{\phi = 2x^2y - xz^3} \Rightarrow \vec\nabla\phi \,;$$

$$\underline{\nabla^2\phi = \vec\nabla\cdot(\vec\nabla\phi)}$$

$$\vec\nabla\phi = (4xy - z^3)\hat i + 2x^2\hat j + (-3xz^2)\hat k$$

$$\vec\nabla\cdot(\vec\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$\underline{= 4y - 6xz}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}(4xy - z^3) = 4y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial y}(2x^2) = 0$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial}{\partial z}(-3xz^2) = 6xz$$

Νά αποδειχθούν οί ταυτότητες

α) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$ β) $\vec{\nabla}\times(\vec{\nabla}\phi)=0$

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\overbrace{\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)}^{(\nabla\times A)_x}\hat{i}+\overbrace{\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)}^{(\nabla\times A)_y}\hat{j}+$$

$$+\underbrace{\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial z}\right)}_{(\nabla\times A)_z}\hat{k}$$

$$\vec{\nabla}\cdot\left[(\vec{\nabla}\times\vec{A})\right]=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+$$

$$+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial y}\right)=\cancel{\frac{\partial^2 A_z}{\partial x\partial y}}-\cancel{\frac{\partial^2 A_y}{\partial x\partial z}}$$

$$+\cancel{\frac{\partial^2 A_z}{\partial y\partial z}}-\cancel{\frac{\partial^2 A_z}{\partial y\partial x}}$$

$$+\cancel{\frac{\partial^2 A_y}{\partial z\partial x}}-\cancel{\frac{\partial^2 A_x}{\partial z\partial y}}$$

$$=0$$

$$\boxed{\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}}$$ νά αποδειχθή

$$\vec{\nabla}\times(\vec{\nabla}\phi)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}=0$$

$$\oint_C \vec{A}\cdot d\vec{\ell}=\int_S \vec{\nabla}\times\vec{A}\cdot d\vec{S}\ , \qquad \oint_S \vec{A}\cdot d\vec{S}=\int_V \vec{\nabla}\cdot\vec{A}\,dV$$

$\vec{A}=\vec{\nabla}\phi$

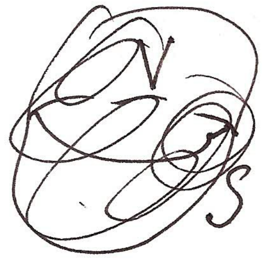
$\oint_C \vec{\nabla}\phi\cdot d\vec{\ell}$

Stokes

Gauss

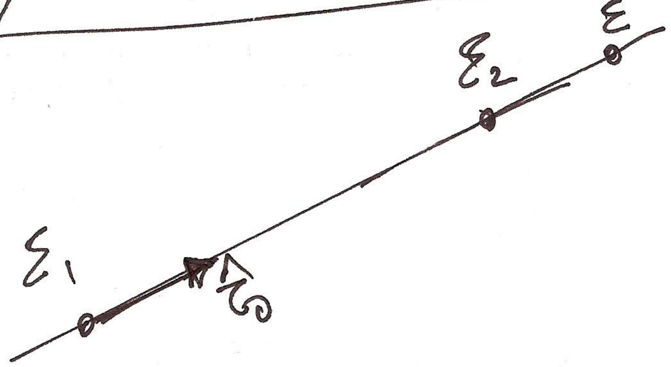
$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$

$C\to 0$



$\vec{\nabla}\times(\vec{\nabla}\phi)=0$

Εξίσωση ευθείας



$$\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|}=\hat{r}_0$$

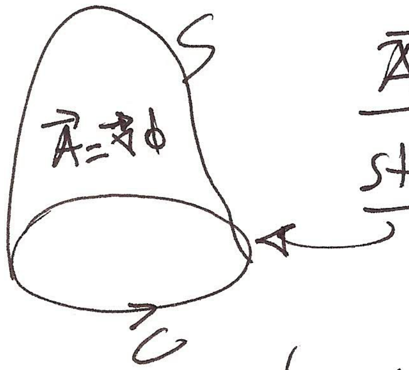
$\Sigma \to \vec{r}(t)$

$\Sigma_1 \to \vec{r}_1$

$\Sigma_2 \to \vec{r}_2$

$$\vec{r}(t)=t\,\hat{r}_0=t\left(\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_2-\vec{r}_1|}\right)$$

α)



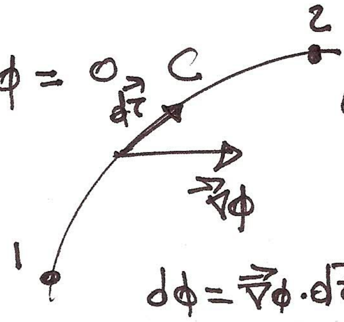
$\vec{A}=\vec{\nabla}\times\phi$

Stokes

$$\oint_C \vec{A}\cdot d\vec{r}=\oint_C \vec{\nabla}\phi\cdot d\vec{r}=\int_S \vec{\nabla}\times(\vec{A})\cdot d\vec{S}=$$

$$=\int_S \vec{\nabla}\times(\vec{\nabla}\phi)\cdot d\vec{S}=0$$

$$\oint_C \vec{\nabla}\phi\cdot d\vec{r}=\oint_C d\phi=0$$

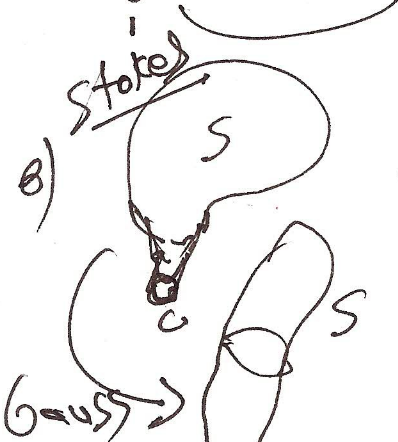


$$\boxed{\vec{\nabla}\times(\vec{\nabla}\phi)=0}$$

$$\int_1^2 d\phi=\phi_2-\phi_1$$

$d\phi=\vec{\nabla}\phi\cdot d\vec{r}$

β)



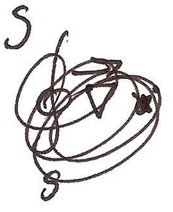
$$\oint_C \vec{A}\cdot d\vec{r}=\int_S \vec{\nabla}\times\vec{A}\cdot d\vec{S}=0 \quad \left(\begin{array}{l}\text{γιά}\\ C\to 0\end{array}\quad \oint_C \vec{A}\cdot d\vec{r}=0\right)$$

$$\oint_S \vec{F}\cdot d\vec{S}=\int_V \vec{\nabla}\cdot\vec{F}\,dV \qquad \text{Gauss}$$

$\vec{F}=\vec{\nabla}\times\vec{A}$

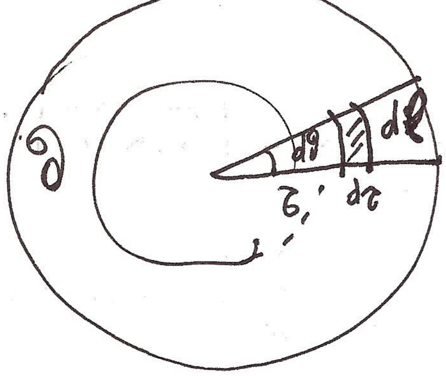
$$\oint_S \vec{\nabla}\times\vec{A}\cdot d\vec{S}=\int_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\,dV=0$$

$C\to 0$



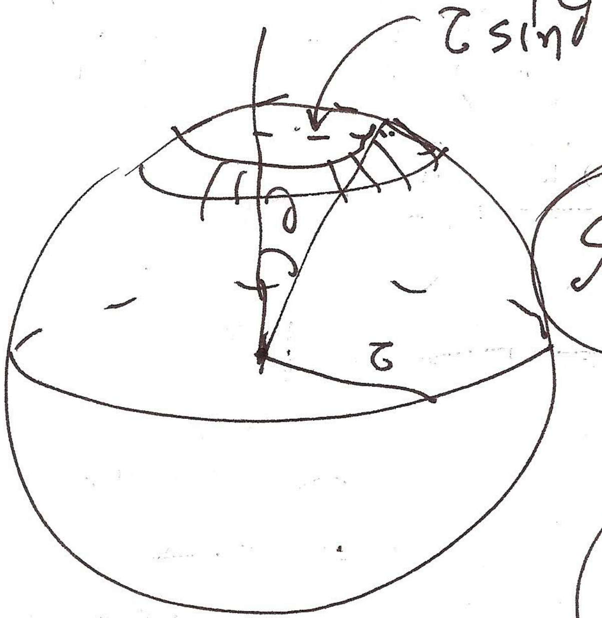
$$\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0}$$

εμβαδόν κύκλου

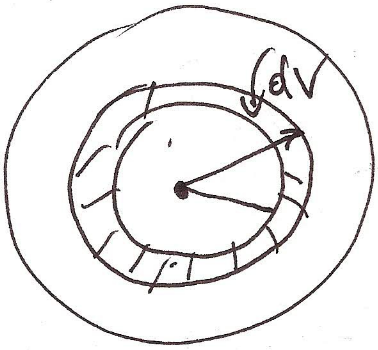


$$ds = dr \cdot (dl)$$
$$= dr \cdot (r\,d\theta) = r\,d\theta\,dr$$
$$S = \int ds = \int_\theta \int_r r\,d\theta\,dr =$$
$$= \int_r r \left(\int_\theta d\theta \right) dr$$
$$= \int_r r\,(2\pi)\,dr = 2\pi \int_0^r r\,dr$$
$$= 2\pi \left(\tfrac{1}{2} r^2 \Big|_0^r \right)$$
$$= \pi r^2$$

$$\int d\theta = 2\pi$$



$$S = 4\pi r^2$$



$$\int_0^a 4\pi r^2\,dr = \int dV$$
$$4\pi \int r^2\,dr = \frac{4\pi r^3}{3}$$

$$V = \frac{4\pi}{3} r^3$$

$$\phi(x,y,z) \xrightarrow{\vec{\nabla}} \vec{\nabla}\phi(x,y,z) \xrightarrow{\vec{\nabla}\cdot} \vec{\nabla}\cdot(\vec{\nabla}\phi)$$

$$\vec{F}(x,y,z) \xrightarrow{\vec{\nabla}\cdot} \vec{\nabla}\cdot\vec{F}\,, \qquad \vec{F}(x,y,z) \xrightarrow{\vec{\nabla}\times} \vec{\nabla}\times\vec{F}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

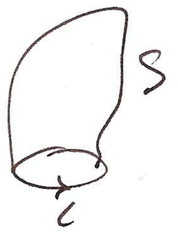
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$\nabla^2\phi =$ Λαπλασιανή

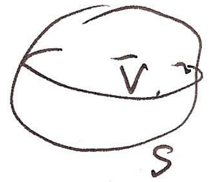
$\vec{\nabla}\cdot\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$ (βαθμίδα)

$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial}{\partial y}F_y + \frac{\partial F_z}{\partial z}$ απόκλιση

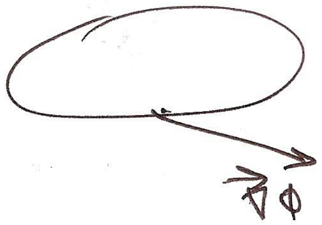
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \text{στροφή}$$



$$\Sigma_F = \oint_C \vec{F}\cdot d\vec{\ell} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} \quad \text{(Stokes)}$$



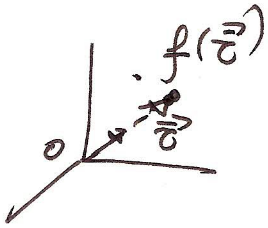
$$\oint_F = \oint_S \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\,dV \quad \text{(Gauss)}$$



$$\int \vec{\nabla}\phi\cdot d\vec{\ell} = \int d\phi = \Delta\phi$$

$$\oint \vec{\nabla}\phi\cdot d\vec{\ell} = 0$$

Θεμελιώδεις τανυστικές σχέσεις



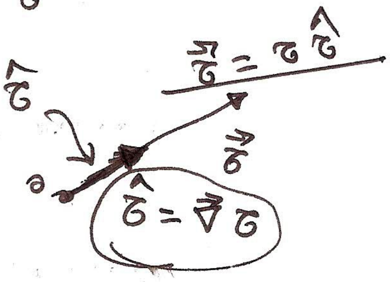
$$f(\vec r) = f(x\hat i + y\hat j + z\hat k) = f(x,y,z)$$

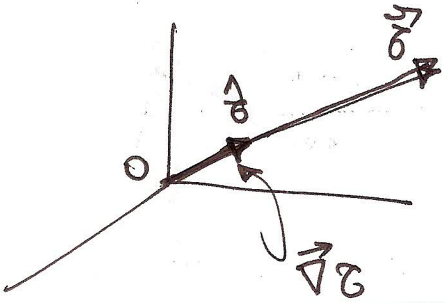
$$\vec\nabla f = \frac{\partial f}{\partial x}\hat i + \frac{\partial f}{\partial y}\hat j + \frac{\partial f}{\partial z}\hat k$$

$$r = (x^2+y^2+z^2)^{1/2} = f(x,y,z) \qquad \vec\nabla r = \frac{\partial r}{\partial x}\hat i + \frac{\partial r}{\partial y}\hat j + \frac{\partial r}{\partial z}\hat k$$

$$\vec\nabla r = \frac{x}{r}\hat i + \frac{y}{r}\hat j + \frac{z}{r}\hat k = \frac{x\hat i + y\hat j + z\hat k}{r} = \frac{\vec r}{r} = \hat r$$

$$\boxed{\vec\nabla r = \frac{\vec r}{r} = \hat r}$$





$$\hat r = \frac{\vec r}{r} = \vec\nabla r$$

$$\boxed{\vec\nabla g(r) = \frac{dg(r)}{dr}\hat r = \frac{dg}{dr}\vec\nabla r}$$

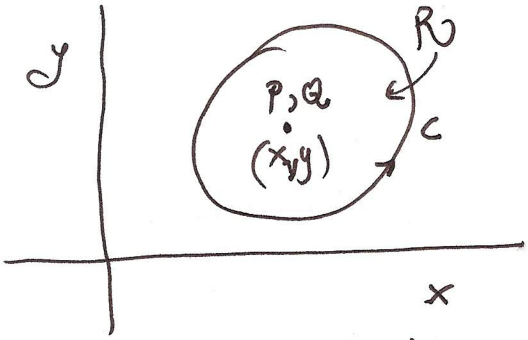
$$g\left[(x^2+y^2+z^2)^{1/2}\right] = g(r)$$

$$g(r) = r^n \Longrightarrow \vec\nabla g = \frac{dg}{dr}\vec\nabla r = (n r^{n-1})\hat r = (n r^{n-1}\hat r)$$

$$\vec\nabla\left(\frac{1}{r}\right) = \vec\nabla(r^{-1}) = (-1)\, r^{-2}\hat r = -\frac{\hat r}{r^2} \qquad (r \neq 0)$$

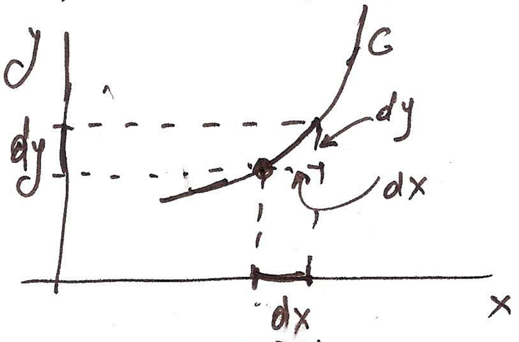
$$\nabla^2\left(\frac{1}{r}\right) = \vec\nabla\cdot(\vec\nabla r) = 0 \qquad (r \neq 0)$$

Θεώρημα Green

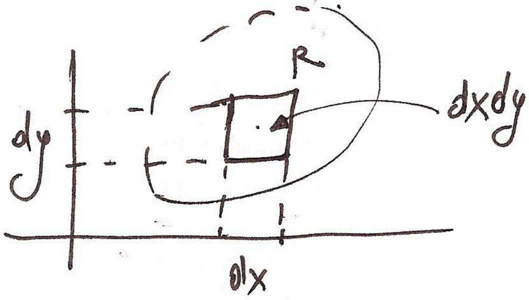


$$P(x,y),\ Q(x,y)$$

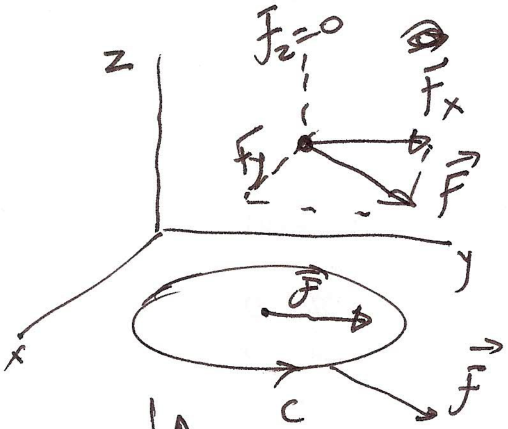
$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$



$\oint_C P\,dx + Q\,dy =$ επικαμπύλιο ολ.



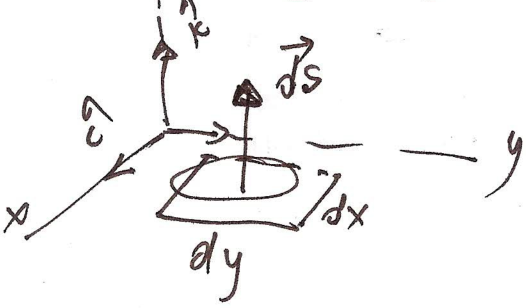
$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =$ επιφανειακό ολ.



$$\boxed{\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} + 0\,\hat{z}}$$

$F_x = P(x,y,z),\ F_y = Q(x,y,z),\ F_z = 0$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}\ ,$$

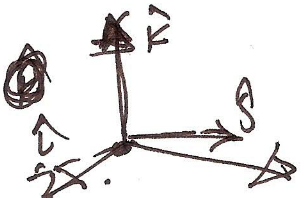


$\vec{dS} = (dx\,dy)\,\hat{k}$

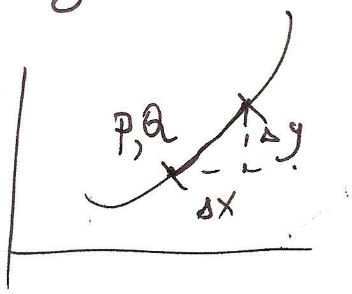
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x , & F_y , & F_z \end{vmatrix} =$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} , & \hat{j} , & \hat{k} \\ \frac{\partial}{\partial x} , & \frac{\partial}{\partial y} , & \frac{\partial}{\partial z} \\ P , & Q , & 0 \end{vmatrix} = 0\,\hat{i} + 0\,\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\vec{\nabla}\times\vec{F}\cdot\vec{dS} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

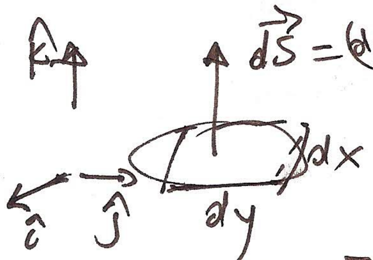


$$\vec{F}\cdot d\vec{r} = F_x\,dx + F_y\,dy + F_z\,dz$$
$$= P\,dx + Q\,dy + 0\,dz = P\,dx + Q\,dy$$
$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \sum_C (P\,\Delta x + Q\,\Delta y)$$



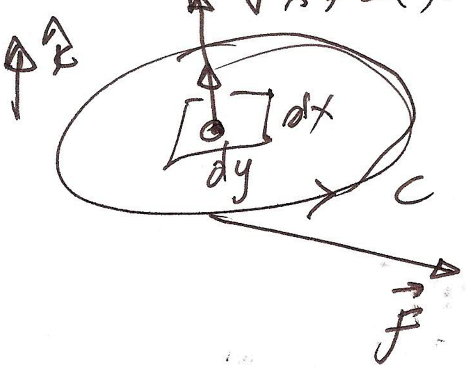
$$\oint_C \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$
$$\oint_C P\,dx + Q\,dy = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} =$$

$$\vec{\nabla}\times\vec{F}\cdot d\vec{S} = \left[0\,\hat{i} + 0\,\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}\right]\cdot\left[0\,\hat{i} + 0\,\hat{j} + (dx\,dy)\,\hat{k}\right]$$



$$d\vec{S} = (dx\,dy)\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + (dx\,dy)\,\hat{k}$$
$$\vec{\nabla}\times\vec{F}\cdot d\vec{S} = 0 + 0 + \left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$
$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

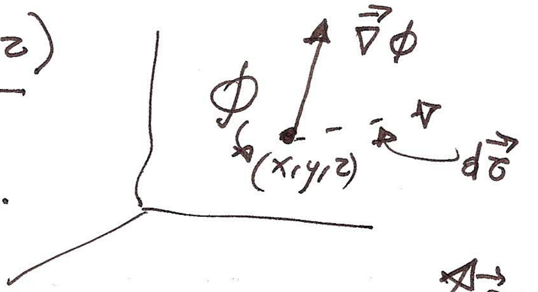
$$\vec{\nabla}\times\vec{F} = (\;)\,\hat{k}$$



$$\boxed{\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy}$$
$$\oint \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S}$$

Φυσικό νόημα (∇, $\vec{\nabla}\cdot$, $\vec{\nabla}\times$)

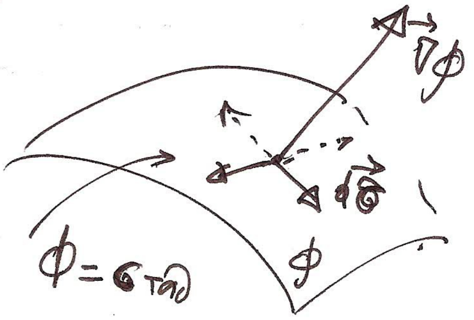
α) $(\vec{\nabla})$: $\phi(x,y,z) \longrightarrow \underbrace{\vec{\nabla}\phi(x,y,z)}_{\text{βαθμίδα}}$



$\vec{\nabla}\phi \perp$ ισοφασική επιφάνεια $\phi=$ σταθ.

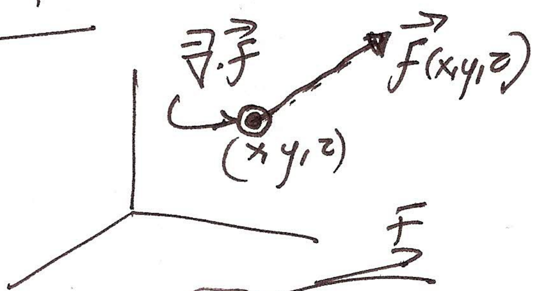
$\phi(x,y,z)=$ σταθ. $\Rightarrow$ (ισο-επιφάνεια ϕ)

$$\boxed{d\phi = \vec{\nabla}\phi\cdot d\vec{r}}$$



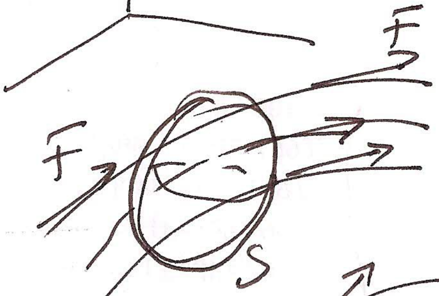
β) $(\vec{\nabla}\cdot)$: $\vec{F}(x,y,z) \longrightarrow \vec{\nabla}\cdot\vec{F}(x,y,z) =$ βαθμωτό

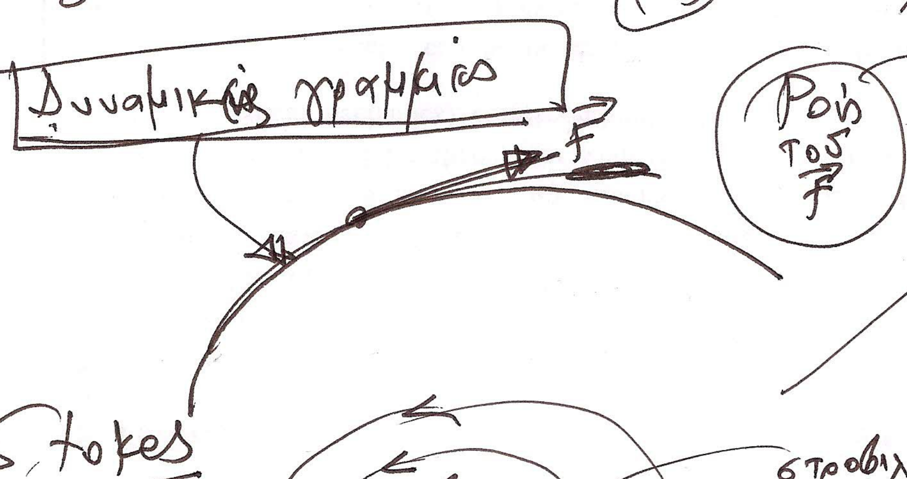
$\vec{\nabla}\cdot\vec{F} =$ απόκλιση $\vec{F}$



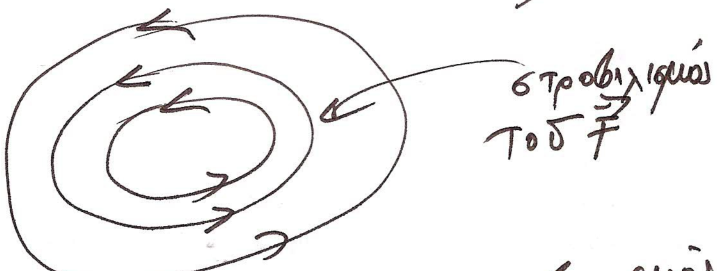
Θεώρημα Gauss

$$\Phi_F = \oint_S \vec{F}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \text{Ροή γραμμών } (\vec{F})$$

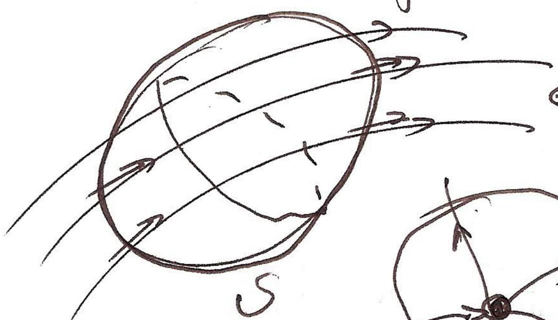




Stokes

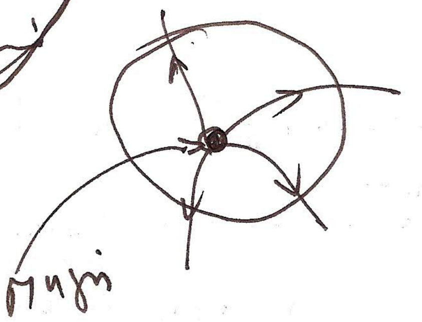


$$\Gamma_F \equiv \oint_C \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \text{στροβιλισμός του } \vec{F}$$



$\vec{\nabla}\cdot\vec{F} = 0 \Rightarrow$ (Ροή $= 0$)

$\Phi_F = 0$



$\vec{\nabla}\cdot\vec{F} \neq 0$ (πηγές, $\vec{\nabla}\cdot\vec{F} \neq 0$)

$\Phi_F \neq 0$

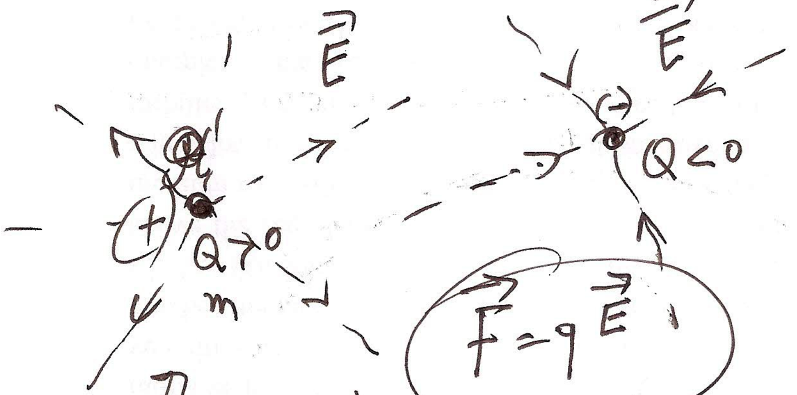
αστρόβιλο πεδίο

$\vec{\nabla}\times\vec{F} = 0$

$\vec{\nabla}\times\vec{F} \neq 0$

(στροβιλισμός)

(στροβιλό πεδίο)



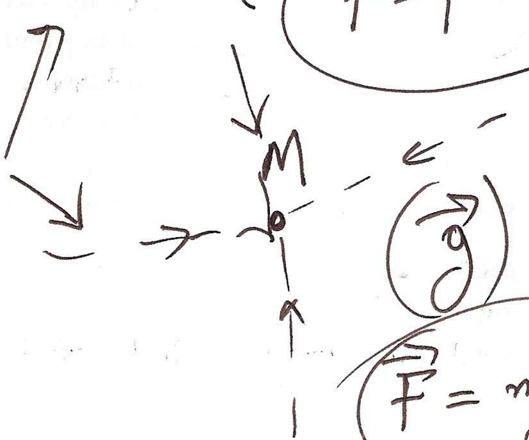
ηλεκτρικό πεδίο Coulomb

$\vec{E}$

$\vec{F} = q\vec{E}$

βαρυτικό πεδίο $\vec{g}$

(αστρόβιλα πεδία)



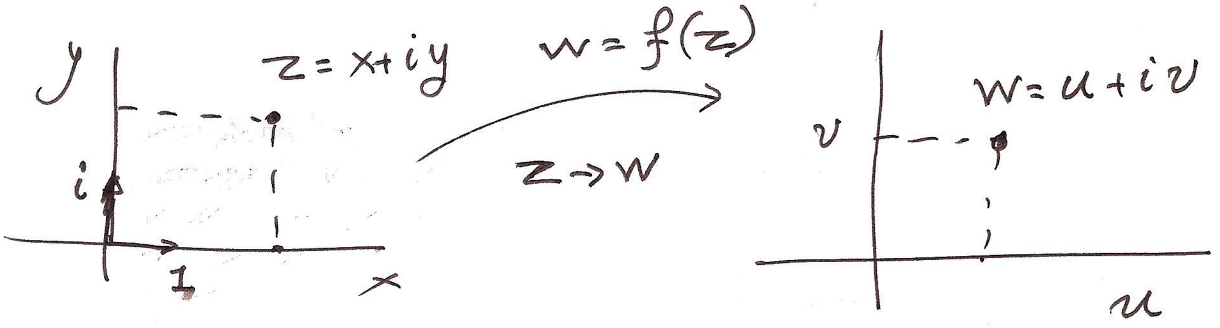
$\vec{F} = m\vec{g}$

$(\vec{E}, \vec{g})$ αστρόβιλα

$\vec{\nabla}\cdot\vec{E} = 0$

$\vec{\nabla}\cdot\vec{g} = 0$

Μιγαδικές Συναρτήσεις



$W = f(z)$

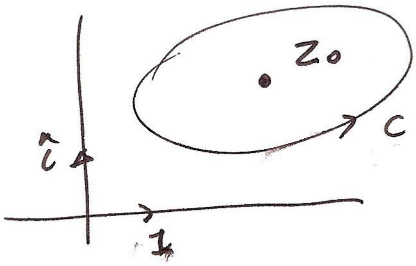
$$\boxed{\begin{aligned}(x,y) &\to u(x,y)\\ (x,y) &\to v(x,y)\end{aligned}}$$

$$\boxed{w = f(z)}$$

$$\boxed{\begin{aligned} z &\to w = u+iv\\ z &\to u = u(z) = u(x,y)\\ z &\to v = v(z) = v(x,y)\end{aligned}}$$

$\underline{f = \text{αναλυτική στο } z_0}$

$$\text{ορ}_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \text{μιγαδικός αριθμός (να υπάρχει το όριο)}$$



$$\int_c f(z)\,dz = \text{ολοκλήρωμα στην (c)}$$

αν $\begin{cases} f(z) \text{ αναλυτική} \\ C = \text{κλειστή} \end{cases}$

$$\oint_c f(z)\,dz = 0$$

Θεώρημα Cauchy $\left(\oint_c f(z)\,dz = \oint = 0\right)$

Θεώρημα Cauchy

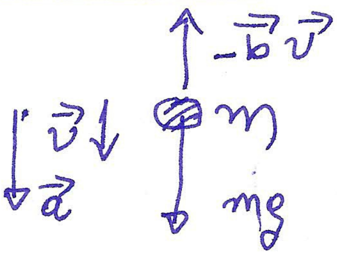
$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-z_0}\,dz}$$

$$\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$$

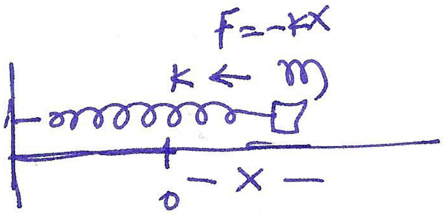
$$\oint_c \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n = 2,3,4 \\ 1,\ (z-a),\ (z-a)^2, \dots & n = 0,-1,-2,\dots \end{cases}$$

Διαφορικές εξισώσεις

$$m\frac{d\vec{v}}{dt} = -mg\hat{k} - b\vec{v} \Rightarrow$$



Πτώση σε αέρα ή υγρό



$$F = -kx$$

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

(Ταλάντωση)

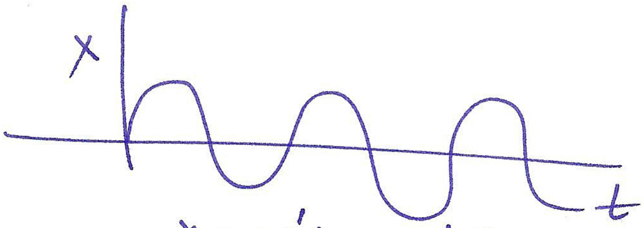
$$\boxed{x = x_0\cos(\omega_0 t+\varphi)}$$

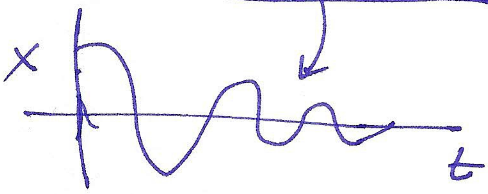
Ταλάντωση με απόσβεση

$$m\frac{d^2x}{dt} = -\lambda v - kx \Rightarrow m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0$$

~~Εξαναγκασμένη Ταλάντωση~~

$$x = x_0 e^{-\gamma}\cos(\omega_0 t+\varphi)$$





Χωρίς απόσβεση

$$x = x_0\cos(\omega_0 t+\varphi)$$
$$\omega_0 = \sqrt{k/m}$$

απόσβεση

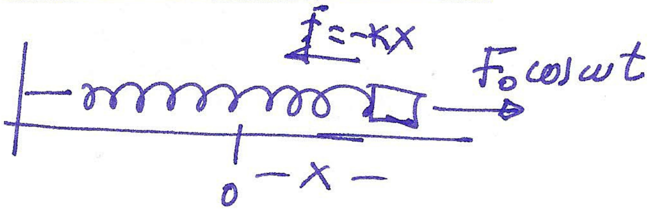
$$x = x_0 e^{-\gamma t}\cos(\omega_0 t+\varphi)$$
$$\lambda = \gamma,\ \omega_0 = \sqrt{k/m}$$

Εξαναγκασμένη Ταλάντωση

$$m\frac{d^2x}{dt^2} = -\lambda v - kx + F_0\cos\omega t \Rightarrow m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

$$x = x_0\sin(\omega t - \varphi)$$

Εξηναγκασμένη ταλάντωση



$$F_0\cos\omega t = m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx$$

a) $$m\frac{d^2x}{dt^2} = -kx - \lambda\frac{dx}{dt} + F_0\cos\omega t$$

$$x = x_0\sin(\omega t - \varphi)$$

$$F_0 = \sqrt{(m\omega^2 x_0 - kx_0)^2 + (\lambda\omega x_0)^2}$$

$$\frac{dx}{dt} = v = \omega x_0\cos(\omega t - \varphi) = v_0\cos(\omega t - \varphi)$$

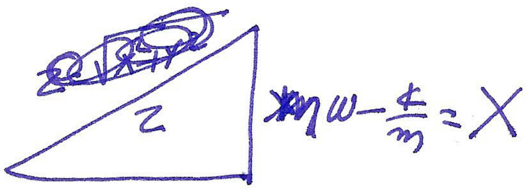
$$F_0 = \omega x_0\sqrt{\left(m\omega - \frac{k}{\omega}\right)^2 + \lambda^2}$$

$$\frac{d^2x}{dt^2} = a = -\omega^2 x_0\sin(\omega t - \varphi) = a_0\sin(\omega t - \varphi)$$

$$F_0 = v_0 Z$$

$$Z = \sqrt{\left(m\omega - \frac{k}{\omega}\right)^2 + \lambda^2}$$

(εμπέδηση)

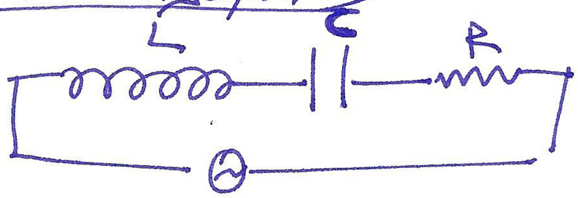


$$Z = \sqrt{X^2 + R^2}$$

$$X = \sqrt{\left(m\omega - \frac{k}{\omega}\right)^2} = \left(m\omega - \frac{k}{\omega}\right)$$

$$R = \lambda$$

Ταλάντωση κυκλώματος



$$V_C = \frac{q}{C} \qquad V_L = L\frac{dI}{dt}, \quad V_R = IR$$

$$V = F_0\cos\omega t = V_0\cos\omega t$$

$$\oint \vec{E}\cdot d\vec{l} = IR$$ (Νόμος Ohm, $\vec{E}$ = ηλεκτρικό πεδίο)

b) $$L\frac{d^2q}{dt^2} + R\frac{dI}{dt} + \frac{q}{C} = V_0\cos\omega t$$

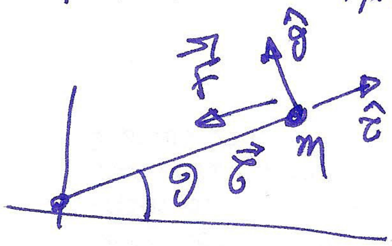
$$I = \frac{dq}{dt}, \quad \frac{dI}{dt} = \frac{d^2q}{dt^2}$$

$$(a,b) \rightarrow \left(m \leftrightarrow L,\ \lambda \leftrightarrow R,\ k = \frac{1}{C},\ F_0 \rightarrow V_0\right)$$

$$V_0 = \omega q_0\sqrt{\left(L\omega - \frac{1}{\omega C}\right)^2 + R^2} = I_0\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}$$

$$V_0 = I_0 Z \qquad Z = \sqrt{R^2 + X^2}, \quad X = \omega L - \frac{1}{\omega C}$$

Κεντρική δύναμη - κεντρική κίνηση



$$\vec F = f(r)\,\hat r$$

$$m\frac{d\vec v}{dt} = f(r)\,\hat r$$

$$\begin{cases} \vec v = \dot r\,\hat r + r\dot\theta\,\hat\theta \\ \vec a = (\ddot r - r\dot\theta^2)\,\hat r + (r\ddot\theta + 2\dot r\dot\theta)\,\hat\theta \end{cases}$$

$$\vec F = m\frac{d\vec v}{dt}$$

$$\begin{cases} m a_r = F_r = m\dfrac{dv_r}{dt} \\ m a_\theta = F_\theta = m\dfrac{dv_\theta}{dt} \end{cases} \Rightarrow \begin{cases} m(\ddot r - r\dot\theta^2) = f(r) \\ m(r\ddot\theta + 2\dot r\dot\theta) = 0 \end{cases}$$

$$\vec h = \vec r \times \vec v = \frac{\vec L}{m} = r^2\dot\theta\,\hat k$$

$$\frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - \int f(r)\,dr = E \quad (E = E_\kappa + E_\delta)$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2(E - V)/mh^2, \quad u = \frac{1}{r}$$

~~κίνηση~~ κίνηση πλανητών
δορυφόρων

$$f(r) = -k/r^2$$

d

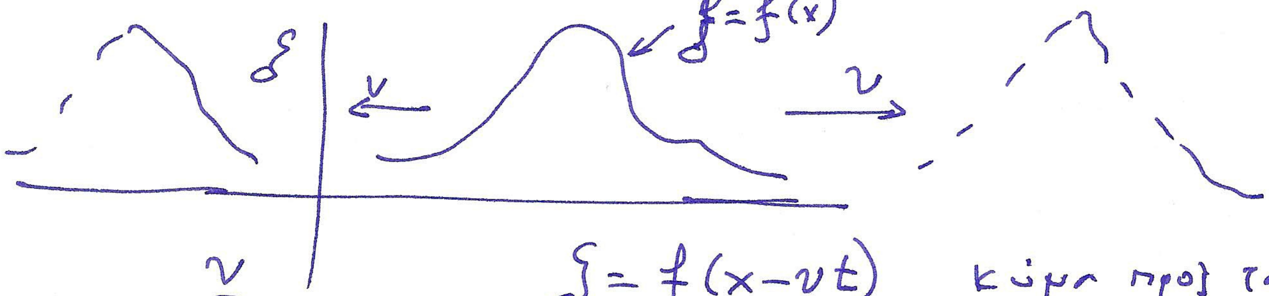
$$\frac{d^2u}{d\theta^2} + u = -\frac{f(u)}{mh^2u^2} \Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}}$$

$$\left(u = \frac{1}{r}\right)$$

$$u = A\cos\theta + B\sin\theta + k/mh^2 \Rightarrow$$

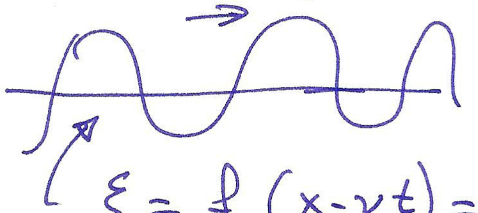
$$\frac{1}{r} = u \Rightarrow \boxed{r = \frac{1}{k/mh^2 + C\cos\theta}} = \frac{p}{1 + \varepsilon\cos\theta}$$

Κύματα



$$\xi = f(x - vt) \qquad \text{κύμα προς τα δεξιά}$$

$$\xi = f(x + vt) \qquad \text{κύμα προς τα αριστερά}$$



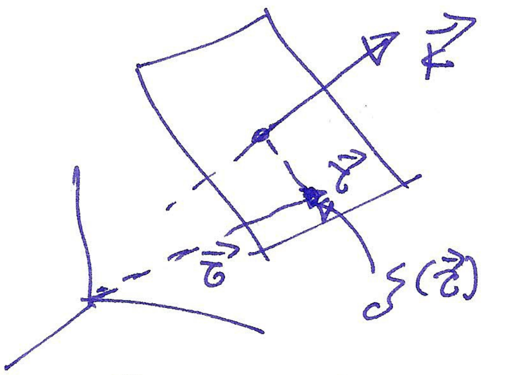
$$\xi = f(x - vt) = \xi_0 \sin(kx - \omega t) \qquad \begin{cases} k = \dfrac{2\pi}{\lambda} \\ \omega = \dfrac{2\pi}{T} \\ v = \dfrac{\omega}{k} \end{cases}$$

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$

$$\nabla^2 \xi - \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} = 0 \qquad \xi = \xi_0\, \eta\mu(\vec{k}\cdot\vec{r} - \omega t)$$

$$\vec{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$$

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} = 0$$



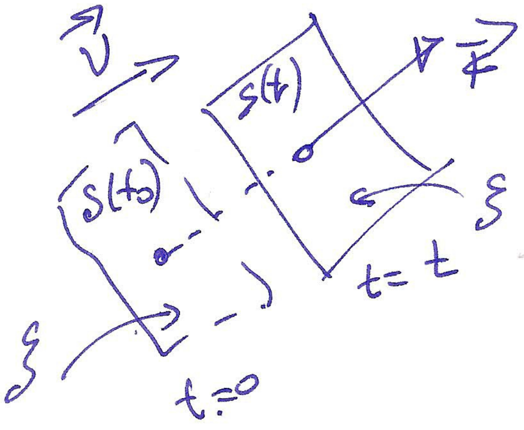
$$\xi = f(\vec{k}\cdot\vec{r} - \omega t)$$

$\vec{k}$ = κυματοδιάνυσμα

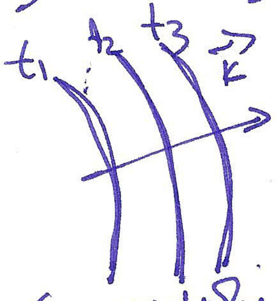
$\vec{\nabla}\xi \perp$ ισοφασική επιφάνεια (S)

$$\xi = f(\vec{k}\cdot\vec{r} - \omega t) = \sigma\tau\alpha\theta \Rightarrow$$

$$\vec{k}\cdot\vec{r} - \omega t = \sigma\tau\alpha\theta$$



$$\xi(t=0) = \xi(t=t, \vec{r} = \vec{v}\, t)$$



σφαιρικό κύμα