Solutions of Boussinesq Equation in Semiinfinite Flow Region

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Abstract: In a semiinfinite flow region, prediction of the water table profile due to an abrupt rise or drop in the canal or drain water level in the cases of recharging and discharging aquifers has been done for times equal to 1.0, 2.0, 3.0, 4.0, and 5.0 days by employing a numerical solution and five analytical solutions. Comparison of the water table profile predicted by the proposed numerical solution with the existing analytical solutions (based on L2 and Tchebycheff norms) shows that the performance of Polubarinova-Kochina's 1948 solution is the best, followed by Lockington's 1997 solution, Verigin's 1949 solution, Polubarinova-Kochina's 1949 solution, and Edelman's 1947 solution for both recharging and discharging aquifers. However, for the example considered in this study, for practical purposes, any of these solutions except the Edelman solution may be adopted for predicting water table heights, because the maximum relative percentage difference in water table heights predicted by these analytical solutions and the proposed numerical solution is not more than ±1.5%.

INTRODUCTION

The Boussinesq equation (1904) was derived using Dupuit's assumptions implying that the inertial forces are negligible and that horizontal components of the velocities do not change with height, thus are functions of only two coordinates, x and y, and of time, t.

The boundary value problem for one-dimensional flow for a semiinfinite unconfined aquifer replenished or drained by a canal or drain may be represented as

$$\frac{\partial h}{\partial t} = \frac{K}{S}\frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right] \tag{1}$$

$$h = h_1, \quad x = 0, \quad t > 0 \tag{2}$$

$$h = h_0, \quad x > 0, \quad t = 0 \tag{3}$$

$$h = h_0, \quad x \to \infty, \quad t > 0 \tag{4}$$

where h [L] = piezometric head; x [L] = horizontal space coordinate; t [T] = time; K [L/T] = hydraulic conductivity, and S = specific yield. The heads h_1 and h_0 are water levels in the canal/drain at $x = 0$ and in the aquifer at $x = \infty$. When the aquifer is being recharged, $h_1 > h_0$, and when the aquifer is being discharged, $h_1 < h_0$.

A number of investigators have studied this boundary value problem and obtained analytical solutions using various approaches. Such studies include those by Edelman (1947), Polubarinova-Kochina (1948, 1949), Verigin (1949), and recently, by Lockington (1997). For minor differences in boundary conditions, solutions of similar form have also been reported by Carslaw and Jaeger (1959), Hantush (1964), Aravin and Numerov (1965), and Bear (1979). Only a few numerical solutions seem to have been developed. The objective of the present study was to obtain a numerical solution of a nonlinear Boussinesq equation for such a semiinfinite flow and to compare it with some of the aforementioned analytical solutions for their accuracy.

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Analytical Solutions Considered for Comparison

A brief description of the analytical solutions used in this study is presented here for ready reference.

Edelman (1947) studied a problem in which the water level in the ditch is lowered or raised, instantaneously causing water to flow out of or into the adjoining aquifer until the water table and the ditch levels are again in equilibrium. The influence of the change in the water level in the ditch was described by the linearized Boussinesq equation by neglecting the term $(\partial h/\partial x)^2$ given by

$$\frac{\partial h}{\partial t} = \frac{KD}{S}\frac{\partial^2 h}{\partial x^2} \tag{5}$$

with the assumption that the difference in head $|h_1 - h_0| << D$, so that D, the depth of flow, is not affected appreciably by the rising water table and the flow in the aquifer is horizontal. Here D was considered equal to h_0. For the initial and boundary conditions defined earlier by (2)–(4), Edelman (1947) obtained the solution of (5) as follows:

$$h = h_0 + (h_1 - h_0)\left[1 - \phi\left(\frac{x}{2\sqrt{KDt/S}}\right)\right] \tag{6}$$

Here, $\phi(u)$ is the probability integral or error function, defined as

$$\phi(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-u_1^2}\,du_1 \tag{7}$$

for which Edelman provided tables, which are also available in standard texts.

Polubarinova-Kochina (1948) applied the Boltzman transformation to the Boussinesq equation and obtained a nonlinear ordinary differential equation as follows:

$$\frac{d^2h^2}{d\eta^2} + \frac{S\alpha^2\eta}{K}\frac{dh}{d\eta} = 0 \tag{8}$$

Transforming (8) with dimensionless variables

$$h = h_1 u, \quad \eta = \frac{x\sqrt{S}}{2\sqrt{kh_1 t}}, \quad \alpha = 2\sqrt{\frac{Kh_1}{S}} \tag{9}$$

yields

$$\frac{d^2u^2}{d\eta^2} + 4\eta\frac{du}{d\eta} = 0 \tag{10}$$

Polubarinova-Kochina expressed its solution in the form of a power series. For a recharging aquifer, the solution is

$$u = 1 + lu_1 + l^2u_2 + l^3u_3 + \cdots \quad (11)$$

where

$$l = \frac{h_0 - h_1}{h_1} \quad (12)$$

She reported the expressions for the coefficients u_1, u_2, and u_3 of this series as a function of η, through which u can be computed. She also tabulated the values of u_1, u_2, and u_3 corresponding to various values of η. Because values of error function u_1 are already available in textbooks, only values of u_2 and u_3 corresponding to the various values of η given by Polubarinova-Kochina (1948) are reproduced in Table 1. Multiplying u by h_1 yields water table height as a function of time and space coordinates. For a discharging aquifer, the solution given was

$$v = 1 + \mu w_1 + \mu^2 w_2 + \cdots \quad (13)$$

where

$$v = u^2 = (h/h_1)^2 \quad (14)$$

$$\mu = \frac{h_1^2 - h_0^2}{h_0^2} \quad (15)$$

and

$$w_1(\eta) = 1 - \phi(\eta) = v_1 \quad (16)$$

$$w_2(\eta) = \frac{1}{2\pi}(1 - e^{-2\eta^2}) - \frac{1}{2\sqrt{\pi}}\eta e^{-\eta^2}v_1 - \frac{1}{2\pi}v_1 + \frac{1}{2\sqrt{\pi}}\eta e^{-\eta^2} \quad (17)$$

The value of h can be computed by taking the square root of v and multiplying it by h_1.

Polubarinova-Kochina (1949) further examined the problem of water seeping from a canal or horizontal bed rock. She considered a linearized form of the Boussinesq equation, neglecting $(\partial h/\partial x)^2$ and considering h, the coefficient of $\partial^2 h/\partial x^2$, approximately constant and equal to D_{av}. She suggested that linearization is permissible if one wants to obtain crude results as a guidance. The linearized differential equation considered was

$$\frac{\partial h}{\partial t} = \frac{KD_{av}}{S}\frac{\partial^2 h}{\partial x^2} \quad (18)$$

where D_{av} = average depth of seepage water, i.e., $(h_0 + h_1)/2$. For the initial and boundary conditions, (2)–(4), she obtained a solution based on the theory of heat conduction, which is similar to (6) except that in this case, D_{av} is taken as $(h_1 + h_0)/2$ in place of h_0, as considered in the Edelman (1947) solution.

For the solution of this boundary value problem, another method of linearization as used by Verigin (1949), i.e., $h^2 = P$, can be used. This transforms the boundary value problem to exactly the same linearized form of the problem as given by (5). Accordingly, the form of the solution remains the same, giving results expressed in the form of h^2.

Lockington (1997) presented the analytical solution for the boundary value problem defined by (1) and for initial and boundary conditions (2)–(4) as given below.

In the case of a recharging aquifer, the solution is

$$h = h_0 + (h_1 - h_0)\left(1 - \frac{x}{\lambda}\sqrt{\frac{S}{Kt}}\right)^{1/\mu} \quad (19)$$

and in the case of a discharging aquifer, it is

$$h = h_0 - (h_0 - h_1)\left(1 + \frac{x}{\lambda}\sqrt{\frac{S}{Kt}}\right)^{-(1/\mu)} \quad (20)$$

Parameters λ and μ are described both for recharging as well as for discharging aquifers in the main text.

TABLE 1. Values of Coefficients u_2 and u_3 Corresponding to Various Values of η

η (1)	u_2 (2)	u_3 (3)
0.0	0.0	0.0
0.1	+0.0141	−0.0039
0.2	+0.0160	−0.0081
0.3	+0.0073	−0.0090
0.4	−0.0092	−0.0049
0.5	−0.0300	+0.0039
0.6	−0.0519	+0.0159
0.7	−0.0718	+0.0280
0.8	−0.0874	+0.0373
0.9	−0.0975	+0.0422
1.0	−0.1017	+0.0418
1.1	−0.1004	+0.0368
1.2	−0.0946	+0.0281
1.3	−0.0855	+0.0194
1.4	−0.0744	+0.0078
1.5	−0.0626	−0.0011
1.6	−0.0510	−0.0079
1.7	−0.0394	−0.0125
1.8	−0.0310	−0.0147
1.9	−0.0232	−0.0151
2.0	−0.0169	−0.0141
2.5	−0.0024	−0.0047
3.0	−0.0002	−0.0006
3.5	−0.0000	−0.0001
4.0	−0.0000	−0.0001

Numerical Solution

A proposed numerical solution to this boundary value problem was obtained by employing the Du Fort and Frankel (1953) method, which is an explicit, three time level finite difference technique. Let $h_{m,n}$ denote $h(x_m, t_n)$. The first derivative of h with respect to x and t and the second derivative of h with respect to x may be obtained as

$$\frac{\partial h_{m,n}}{\partial x} = \frac{h_{m+1,n} - h_{m-1,n}}{2\Delta x} + O(\Delta x)^2 \quad (21)$$

$$\frac{\partial h_{m,n}}{\partial t} = \frac{h_{m,n+1} - h_{m,n-1}}{2\Delta t} + O(\Delta t)^2 \quad (22)$$

$$\frac{\partial^2 h_{m,n}}{\partial x^2} = \frac{h_{m+1,n} - 2h_{m,n} + h_{m-1,n}}{(\Delta x)^2} + O(\Delta x)^2 \quad (23)$$

where O denotes order of error.

Replacing $h_{m,n}$ in (23) with the mean of the values $h_{m,n+1}$ and $h_{m,n-1}$ yields

$$\frac{\partial^2 h_{m,n}}{\partial x^2} = \frac{h_{m+1,n} - h_{m,n+1} - h_{m,n-1} + h_{m-1,n}}{(\Delta x)^2} + O(\Delta x)^2 \quad (24)$$

Neglecting the order of the error terms, (1) can be discretized using (21), (22), and (24). Letting the time periods t_{n-1}, t_n, and t_{n+1} be replaced by U, V, and W, the discretized form of (1) may be expressed as

$$\frac{(W_m - U_m)}{2\Delta t} = \frac{K}{S}\left[V_m\left(\frac{V_{m+1} - W_m - U_m + V_{m-1}}{(\Delta x)^2}\right) + \left(\frac{V_{m+1} - V_{m-1}}{2\Delta x}\right)^2\right] \quad (25)$$

or

$$W_m\left[\frac{1}{2\Delta t}+\frac{KV_m}{S(\Delta x)^2}\right]=\frac{U_m}{2\Delta t}+\frac{K}{S}\left[V_m\left(\frac{V_{m+1}-U_m+V_{m-1}}{(\Delta x)^2}\right)+\left(\frac{V_{m+1}-V_{m-1}}{2\Delta x}\right)^2\right] \quad (26)$$

Because (26) requires calculation at three time levels, i.e., U, V, and W, while other explicit methods require two time levels, this scheme is discretized at intermediate time levels using the forward and central difference formula as

$$\frac{(V_m-U_m)}{\Delta t}=\frac{K}{S}\left[U_m\left(\frac{U_{m+1}-2U_m+U_{m-1}}{(\Delta x)^2}\right)+\left(\frac{U_{m+1}-U_{m-1}}{2\Delta x}\right)^2\right] \quad (27)$$

Here, $V_1 = U_1 = h_1$ for $t > 0$.

Putting $i = 2$ in (27) gives

$$V_2=U_2+\frac{K\Delta t}{S}\left[U_2\left(\frac{U_3-2U_2+U_1}{(\Delta x)^2}\right)+\left(\frac{U_3-U_1}{2\Delta x}\right)^2\right] \quad (28)$$

Here, $U_2 = U_3 = h_0$ (initial condition). Use of (28), the initial condition, and values of K and S help in determining the condition on step size ratio, which is $[\Delta t < (0.07\Delta x)^2]$. In the present analysis, the values of Δt and Δx were taken as 0.0025 days and 2.0 m, respectively.

RESULTS AND DISCUSSION

For the purpose of comparative study, the example given by Lockington (1997) was selected. In this example, the flow in a shallow sand aquifer with hydraulic conductivity $K = 20$ m/d and specific yield $S = 0.27$ was considered. The aquifer was considered to be underlain by a horizontal impermeable base taken as datum and initially having a uniform water level elevation, $h_0 = 2$ m. The water level in the adjoining trench was abruptly raised to the elevation $h_1 = 3$ m to provide a recharging aquifer. Similarly, when water level in the aquifer was at an elevation $h_0 = 3$ m and in the canal/trench it was at an elevation $h_1 = 2$ m, it provided a discharging aquifer. The resulting water table profiles in the aquifer for both cases from $t = 1$ to $t = 5$ days were determined by various solutions, and the results were compared.

In order to compare the analytical solutions with numerical solution, L2 and Tchebycheff norms reported by Prenter (1975) were employed to measure the goodness of the analytical solutions with respect to the proposed numerical solution. The two norms used are given as follows:

- L2 norm

$$\left(\frac{1}{\sqrt{L}}\right)\|C_a - C_n\|_2 = \left(\frac{1}{\sqrt{L}}\right)\left[\int_0^L [C_a(x,t)-C_n(x,t)]^2\,dx\right]^{0.5} \quad (29)$$

This norm gives the average distance of the numeric solution, $C_n(x, t)$, from the analytical solutions, $C_a(x, t)$, at time t. Here integration was performed using the trapezoidal rule.

- Tchebycheff norm

$$\|C_a(x,t)-C_n(x,t)\| = \max_{0\le x\le L}|C_a(x,t)-C_n(x,t)| \quad (30)$$

TABLE 2. Comparison of Water Table Heights for $t = 1$ Day as Predicted by Numerical Solution and Various Analytical Solutions for Recharging Aquifer

X (1)	Numerical solution (2)	Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
0.0	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
10.0	2.6368	2.6250	2.5614	2.6367	2.6035	2.6491
20.0	2.3173	2.3349	2.2456	2.3184	2.2989	2.3441
30.0	2.1152	2.1321	2.0815	2.1164	2.1191	2.1437
40.0	2.0299	2.0202	2.0205	2.0293	2.0398	2.0491
50.0	2.0056	—	2.0038	2.0052	2.0095	2.0119
60.0	2.0008	—	2.0005	2.0005	2.0019	2.0023
70.0	2.0001	—	2.0001	1.9995	2.0003	2.0004
80.0	2.0000	—	2.0000	2.0001	2.0000	2.0001

TABLE 3. Comparison of Water Table Heights for $t = 5$ Days as Predicted by Numerical Solution and Various Analytical Solutions for Recharging Aquifer

X (1)	Numerical solution (2)	Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
0.0	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
10.0	2.8382	2.8219	2.7951	2.8377	2.8163	2.8428
20.0	2.6750	2.6606	2.6035	2.6748	2.6423	2.6854
30.0	2.5182	2.5163	2.4358	2.5180	2.4858	2.5356
40.0	2.3767	2.3890	2.2989	2.3770	2.3528	2.4008
50.0	2.2575	2.2791	2.1946	2.2586	2.2456	2.2864
60.0	2.1647	2.1867	2.1191	2.1659	2.1638	2.1952
70.0	2.0983	2.1122	2.0691	2.0996	2.1038	2.1258
80.0	2.0548	2.0560	2.0398	2.0550	2.0632	2.0774
90.0	2.0286	2.0186	2.0197	2.0280	2.0395	2.0488
100.0	2.0139	2.0010	2.0095	2.0183	2.0205	2.0254
110.0	2.0064	—	2.0043	2.0060	2.0106	2.0132
120.0	2.0027	—	2.0019	2.0022	2.0054	2.0067
130.0	2.0011	—	2.0007	2.0009	2.0026	2.0033
140.0	2.0004	—	2.0003	1.9997	2.0011	2.0014
150.0	2.0001	—	2.0002	1.9993	2.0005	2.0006
160.0	2.0000	—	2.0000	1.9997	2.0003	2.0004

TABLE 4. Comparison of Water Table Heights for $t = 1$ Day as Predicted by Numerical Solution and Various Analytical Solutions for Discharging Aquifer

X (1)	Numerical solution (2)	Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
0.0	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
10.0	2.4325	2.4368	2.3646	2.4290	2.3965	2.4459
20.0	2.7159	2.7126	2.6569	2.7150	2.7011	2.7396
30.0	2.8781	2.8724	2.8452	2.8775	2.8809	2.8991
40.0	2.9548	2.9542	2.9422	2.9554	2.9602	2.9667
50.0	2.9864	2.9886	2.9821	2.9863	2.9905	2.9921
60.0	2.9965	2.9987	2.9955	2.9966	2.9981	2.9984
70.0	2.9993	3.0000	2.9991	2.9993	2.9997	2.9997
80.0	2.9999	—	2.9997	2.9999	3.0000	3.0000
90.0	3.0000	—	3.0000	3.0000	3.0000	3.0000

TABLE 5. Comparison of Water Table Heights for $t = 5$ Days as Predicted by Numerical Solution and Various Analytical Solutions for Discharging Aquifer

X (1)	Numerical solution (2)	Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
0.0	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
10.0	2.2116	2.2184	2.1680	2.2093	2.1837	2.2177
20.0	2.3936	2.3992	2.3286	2.3906	2.3577	2.4059
30.0	2.5458	2.5467	2.4755	2.5427	2.5142	2.5634
40.0	2.6683	2.6654	2.6039	2.6663	2.6472	2.6900
50.0	2.7650	2.7592	2.7112	2.7635	2.7544	2.7879
60.0	2.8380	2.8318	2.7969	2.8373	2.8362	2.8603
70.0	2.8922	2.8867	2.8624	2.8917	2.8962	2.9122
80.0	2.9303	2.9270	2.9103	2.9302	2.9368	2.9469
90.0	2.9567	2.9555	2.9438	2.9567	2.9605	2.9669
100.0	2.9740	2.9747	2.9610	2.9702	2.9795	2.9829
110.0	2.9850	2.9870	2.9801	2.9848	2.9894	2.9911
120.0	2.9916	2.9941	2.9891	2.9917	2.9946	2.9955
130.0	2.9955	2.9978	2.9941	2.9955	2.9974	2.9978
140.0	2.9977	2.9994	2.9970	2.9977	2.9989	2.9991
150.0	2.9989	2.9999	2.9985	2.9989	2.9995	2.9996
160.0	2.9995	—	2.9993	2.9995	2.9997	2.9998
170.0	2.9997	—	2.9996	2.9997	2.9998	2.9999
180.0	2.9999	—	2.9998	2.9999	3.0000	3.0000
190.0	2.9999	—	2.9999	2.9999	3.0000	3.0000
200.0	3.0000	—	3.0000	3.0000	3.0000	3.0000

This describes the maximum difference between the analytical and numerical solutions in the flow domain $0 \leq x \leq L$ at time t.

The piezometric head as a function of space and time coordinates computed by employing the proposed numerical solution based on the Du Fort and Frankel (1953) method was compared with the piezometric head obtained by the analytical solutions of Edelman (1947), Polubarinova-Kochina (1948, 1949), Verigin (1949), and Lockington (1997) for recharging and discharging aquifers for each of the first 5 days. The results, however, have been presented here only for recharging and discharging aquifers at $t = 1$ and 5 days, as given in Tables 2–5.

For recharging and discharging aquifers, the relative percentage difference in spatial and temporal variation of the water table heights predicted by the solutions of Edelman (1947), Polubarinova-Kochina (1948, 1949), Verigin (1949), and Lockington (1997) with respect to the proposed numerical solution was computed, and the range of relative percentage difference is given in Table 6. There was no definite trend of variation in the relative percentage difference with respect to time. With respect to space, the water table heights predicted by Edelman (1947) were slightly lower than the proposed numerical solution, whereas the water table heights predicted by Verigin (1949) were slightly higher than the proposed numerical solution both for recharging and discharging aquifers. For the other three analytical solutions, there was no distinct trend with respect to space.

TABLE 6. Range of Relative Percentage Difference of Various Solutions with Respect to Numerical Solution

Solution (1)	Recharging aquifer (2)	Discharging aquifer (3)
Edelman (1947)	0.0–3.33	0.0–2.80
Polubarinova-Kochina (1948)	−0.22–0.04	−0.02–0.14
Polubarinova-Kochina (1949)	−0.57–1.30	−0.22–1.50
Verigin (1949)	−1.41–0.0	−0.85–0.0
Lockington (1997)	−1.05–0.64	−0.31–0.23

The L2 and Tchebycheff norms were computed to compare the numerical solution with the analytical solutions for $t = 1.0$, 2.0, 3.0, 4.0, and 5.0 days; results are given in Tables 7 and 8. Results reveal that, for a discharging aquifer, the values of L2 and Tchebycheff norms were minimum for the Polubarinova-Kochina (1948) solution, with successively increasing values, respectively, for the solutions of Lockington (1997), Verigin (1949), Polubarinova-Kochina (1949), and Edelman (1947).

In the case of recharging aquifers, both L2 and Tchebycheff norms provided a trend similar to that of discharging aquifers, establishing a first and second rank of performance for the solutions of Polubarinova-Kochina (1948) and Lockington

TABLE 7. L2 and Tchebycheff Norms for Recharging Aquifer

Norms (1)	Day (2)	Comparison with Numerical Solution: Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
L2 norm	1	0.0130	0.0388	0.0007	0.0140	0.0162
Tch. norm	1	0.0176	0.0754	0.0012	0.0333	0.0285
L2 norm	2	0.0135	0.0394	0.0009	0.0143	0.0164
Tch. norm	2	0.0201	0.0824	0.0025	0.0313	0.0296
L2 norm	3	0.0142	0.0401	0.0010	0.0147	0.0167
Tch. norm	3	0.0223	0.0798	0.0023	0.0337	0.0294
L2 norm	4	0.0139	0.0402	0.0013	0.0146	0.0166
Tch. norm	4	0.0230	0.0822	0.0040	0.0336	0.0306
L2 norm	5	0.0140	0.0425	0.0013	0.0155	0.0171
Tch. norm	5	0.0220	0.0824	0.0040	0.0327	0.0305

Note: Tch. norm = Tchebycheff norm.

TABLE 8. L2 and Tchebycheff Norms for Discharging Aquifer

Norms (1)	Day (2)	Comparison with Numerical Solution: Lockington (1997) (3)	Edelman (1947) (4)	Polubarinova-Kochina (1948) (5)	Polubarinova-Kochina (1949) (6)	Verigin (1949) (7)
L2 norm	1	0.0032	0.0322	0.0012	0.0132	0.0123
Tch. norm	1	0.0057	0.0679	0.0035	0.0360	0.0237
L2 norm	2	0.0032	0.0323	0.0012	0.0134	0.0120
Tch. norm	2	0.0066	0.0695	0.0032	0.0346	0.0227
L2 norm	3	0.0037	0.0334	0.0014	0.0139	0.0123
Tch. norm	3	0.0069	0.0695	0.0030	0.0342	0.0237
L2 norm	4	0.0037	0.0327	0.0016	0.0137	0.0121
Tch. norm	4	0.0070	0.0693	0.0036	0.0356	0.0228
L2 norm	5	0.0038	0.0378	0.0017	0.0158	0.0121
Tch. norm	5	0.0068	0.0702	0.0038	0.0359	0.0229

Note: Tch. norm = Tchebycheff norm.

(1997), respectively. The two selected norms, however, could not consistently establish the order of performance for the solutions of Polubarinova-Kochina (1949) and Verigin (1949), because the smaller value of L2 norm was observed with the Polubarinova-Kochina (1949) solution and the smaller Tchebycheff norm was observed for the Verigin (1949) solution. Whether one should select the solution on the basis of the smallest L2 norm or the smallest Tchebycheff norm is a moot point in numerical analysis. Since Verigin (1949) takes into account the nonlinear term in the governing differential equation, his solution was ranked at third place and Polubarinova-Kochina's (1949) solution at fourth place. Edelman's (1947) solution was observed to have the lowest rank of performance.

The following reasons may account for the difference in performance of the analytical solutions from the numerical solution. Edelman (1947) assumed that the distance of the water level from the impervious layer is much greater than the head difference. Such a condition is not met in the present example. This may be a possible reason for the lower predicted values of water table heights in comparison with the numerical solution and for the relative percentage difference up to 3.33 percent. In the case $|h_1 - h_0| << D$, the performance of this solution may be expected to improve. Polubarinova-Kochina (1948) gave the solution of the nonlinear ordinary differential equation in terms of the power series. Since she considered the nonlinear terms of the equation to obtain the solution, this may be the reason why her solution yields results closer to the numerical solution. Her solution requires different values of coefficients to be used in the power series. However, expressions and values are provided only for u_1, u_2, and u_3, respectively. If more coefficients were considered, the accuracy of her solution might improve further. This process of computation is slightly difficult in comparison with other methods. Polubarinova-Kochina (1949) obtained a solution after neglecting the nonlinear term $(\partial h/\partial x)^2$; this may be the reason why her solution is inferior to the numerical solution. The method of linearization with $h^2 = P$ used by Verigin (1949) to account for the term $(\partial h/\partial x)^2$ in his solution seems to have improved his results compared to Polubarinova-Kochina (1949). Because in all of these solutions, the average depth of flow was considered, whereas in the numerical solution the actual depth was considered, the numerical solution may be considered more accurate than the other solutions. Lockington (1997) also considered nonlinear terms, obtained the analytical solution using a weighted residual method, and expressed his solution in the form of simple algebraic equations. His solution is thus easy and handy for computation.

CONCLUSIONS

A numerical solution was obtained for the boundary value problem in a semiinfinite flow region using the three time level explicit Du Fort and Frankel (1953) finite difference method. The analytical solutions of Edelman (1947), Polubarinova-Kochina (1948, 1949), Verigin (1949), and Lockington (1997) were compared with a numerical solution for their performance in predicting water table heights. L2 and Tchebycheff norms were used as theoretical tools to rank the performance of various analytical solutions with respect to the numerical solution for t = 1.0, 2.0, 3.0, 4.0, and 5.0 days. According to these norms, in both recharging and discharging aquifers, the solution of Polubarinova-Kochina (1948) predicted water table heights closest to the values obtained by the proposed numerical solution, followed in decreasing order of performance by Lockington (1997), Verigin (1949), Polubarinova-Kochina (1949), and Edelman (1947). For the example considered in this study, the overall range of relative percentage difference for all the analytical solutions except the Edelman (1947) solution with respect to the numerical solution was observed to be always less than $\pm 1.5\%$ for both recharging and discharging aquifers.

It may be thus concluded that, besides the theoretical ranking based on L2 and Tchebycheff norms as given above, for the example studied, for practical purposes all the analytical solutions considered except the Edelman (1947) solution could be used to predict water table heights in semiinfinite aquifers. The Edelman (1947) solution may be used to predict water table heights provided the condition $|h_0 - h_1| << D$ is strictly satisfied and D may be approximated equal to h_0.

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