

QE12. Measurement Theory

- Classic Measurement Theory
- Principles of Quantum Measurements

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Classical Measurement theory

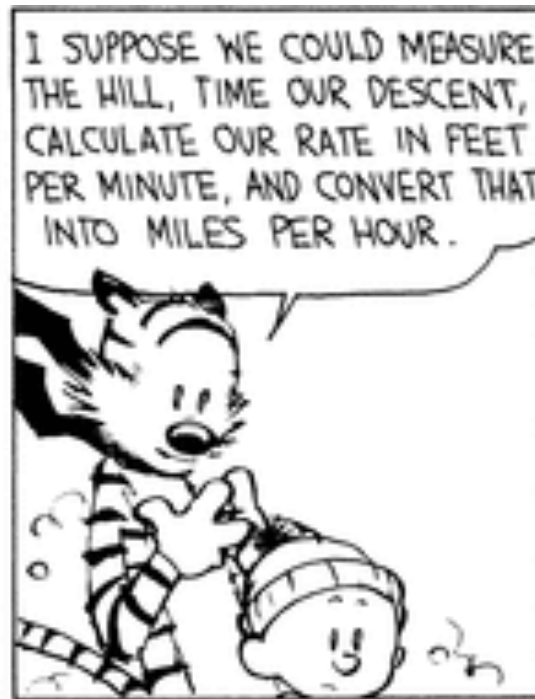
Measurement

(Direct) Measurement : Comparison of a physical quantity with a measurement unit

(Indirect) Measurement of non-physical quantities are related to psychology and sociology etc, such as IQ, level of prosperity, or inflation, are based on human defined quantities.

Measurement is achieved by using special technical means (measuring instruments) and following a specified technical procedure. After the measurement, a number to the measurand is assigned.

The measurement process and/or the measuring instrument may interact with the measurand.

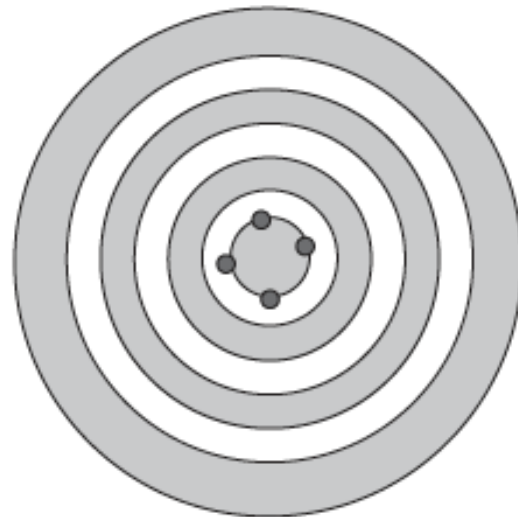


Understanding measurements

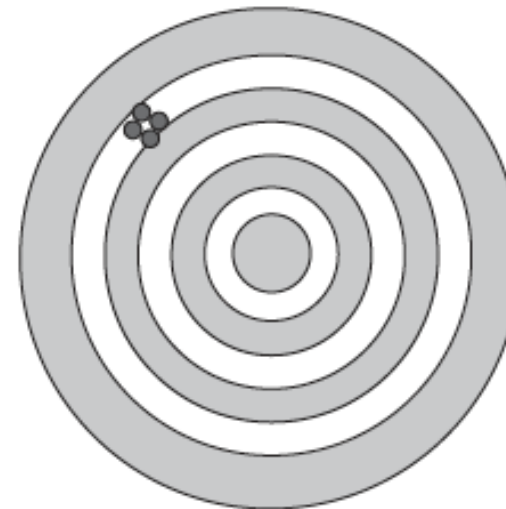
Accuracy is the degree of conformity of a measured or calculated quantity to its actual (true) value.

Repeatability (*precision*) is the degree to which repeated measurements (or calculations) show the same or similar results.

Accuracy and repeatability do not have any innate correlation.



High accuracy,
Low repeatability

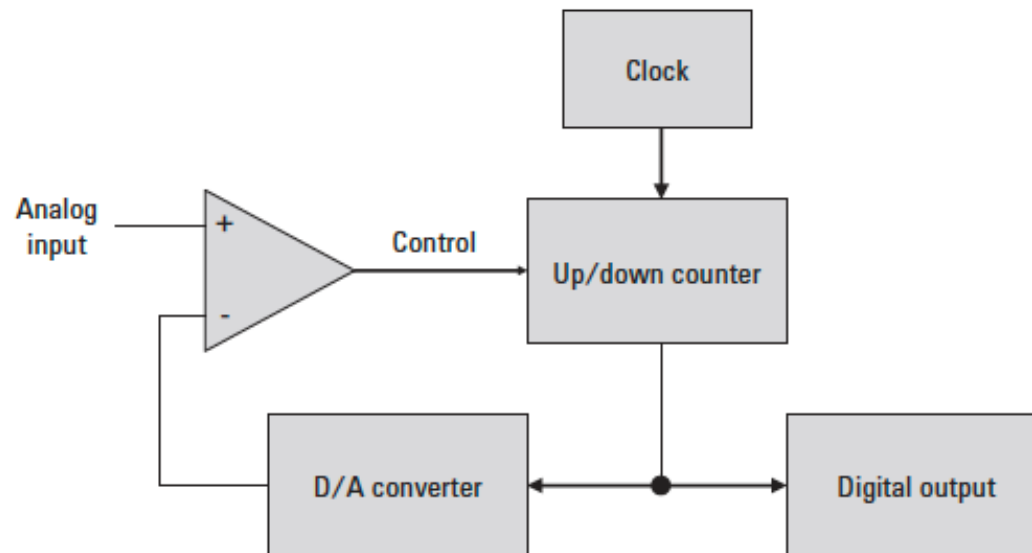


Low accuracy,
High repeatability

Resolution

Resolution is the lowest resolvable quantity of data that an instrument can accurately measure.

The easiest way to think about this is in terms of an analog-to-digital converter (ADC) circuit.



For example, in a 5 V measurement range the ADC measures from -5 V to $+5$ V, which (assuming a 20 bit ADC) yields a readable or resolvable measurement resolution of:

$$\frac{\text{Range}}{\text{\#points}} = \frac{5+5}{2^{20}} = \frac{10}{1.048.576} = 10\mu V$$

Sampling signals – Analog-to-Digital (A/D) conversion

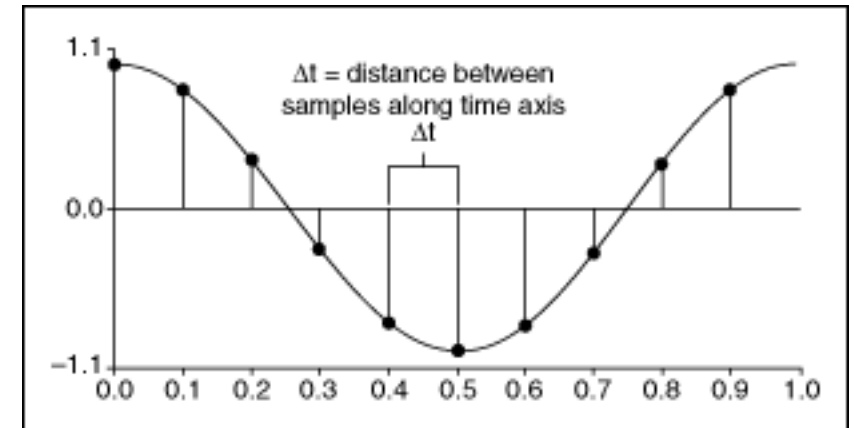
An analog signal consists on an electric signal changing during the time.

When the analog signal is converted into a digital signal, it becomes a list of numbers, or **bits**.

A value measuring is done at regular time intervals.

The **sampling rate (SR)** is the number of times a signal is read in a second (**#samples/s, sampling frequency, F_s**).

As a signal is sampled **n times in a second**, the signal is sampled every **$1/n$ seconds (sampling time, Δt)**



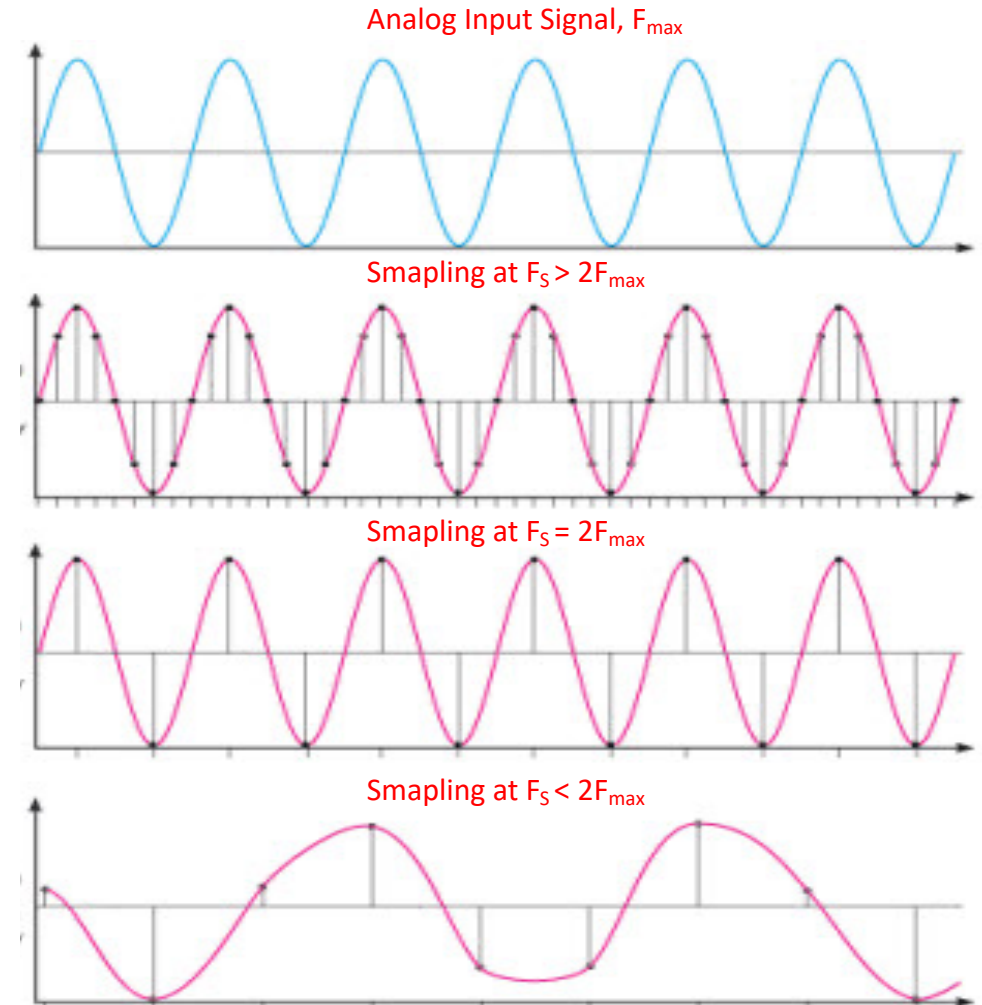
Nyquist-Shannon Theorem

The SR must be equal or superior to the double of the highest frequency of the signal.

$$SR = F_s \geq 2 * F_{max}$$

The Nyquist criterion presumes that the frequency content of the signal has an upper bound, and that the signal's duration has no upper bound.

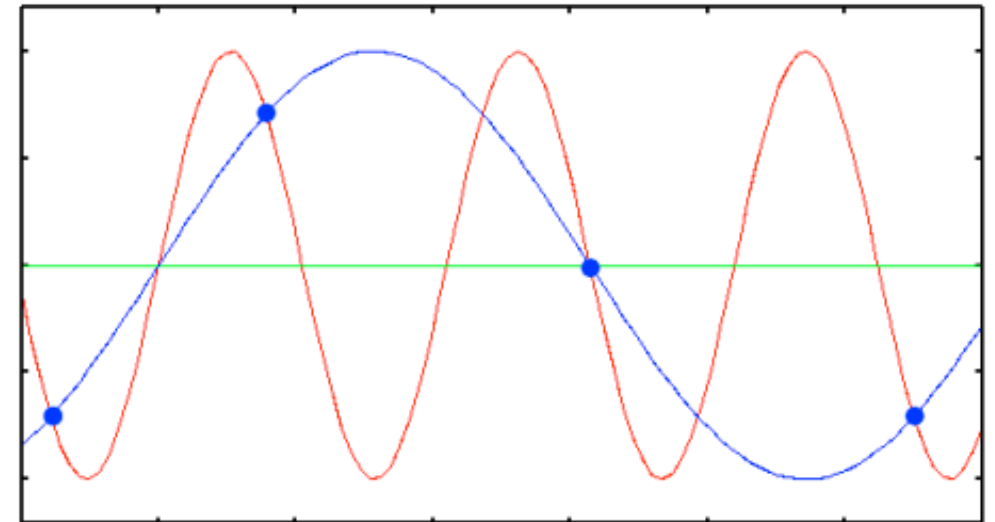
- For a given highest frequency F_{max} , we get the lower bound on the sampling frequency : $2F_{max}$ or **Nyquist rate**. For instance : for a signal whose $F_{max} = 16$ kHz, we need a $F_s = 32$ kHz
- For a given sampling rate, we get the upper bound for frequency components : $B < F_s/2$, or **Nyquist frequency** or F_{max} . For instance : for a signal whose $F_s = 48$ KHz, we can sample signals up to $F_{max} = 24$ KHz.

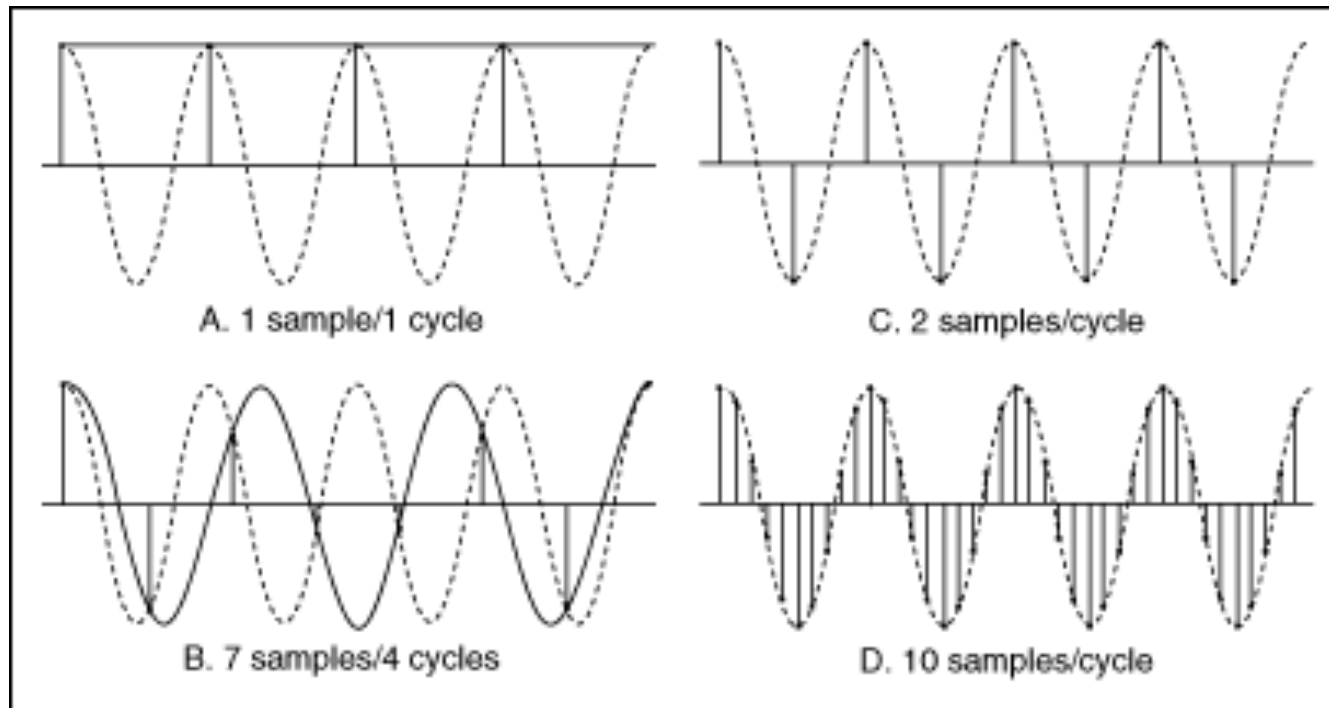


Aliasing

What happens if we sample the signal at a frequency that is lower than the Nyquist rate? When the signal is converted back into a continuous time signal, it will exhibit a phenomenon called **aliasing**. **Aliasing is the presence of unwanted components in the reconstructed signal.**

These components were not present when the original signal was sampled. In addition, some of the frequencies in the original signal may be lost in the reconstructed signal. **Aliasing occurs because signal frequencies can overlap if the sampling frequency is too low.** Frequencies "fold" around half the sampling frequency - which is why this frequency is often referred to as the folding frequency.





Measurement Error – Uncertainty

Any physical value is measured with error/uncertainty; this uncertainty must be defined for any physical value measured or calculated in any experiment.

Every measurement has uncertainty : $L = (24.7 \pm .4)cm$

Systematic uncertainties/errors: these errors appear and stay constant for any repetition of the experiment

Random uncertainties/errors: these errors produce scatter in measured value and may be reduced by repeating the experiments many times

- Obvious errors or outliers are the results which are noticeably inconsistent with the rest of the data set
- Instrumental errors come from a limited resolution of a measuring apparatus
- Statistical errors become important when an experiment is repeated many time in nominally identical conditions, but the outcomes vary within some range around some average value

Statistical analysis is necessary!

Measurement Statistics

Average (Mean) value

$$x_{ave} = \frac{\sum_{i=1}^N x_i}{N}$$

The measurement should be reported as

$$x_{ave} \pm \Delta x$$

Standard error of the mean (SEM)

Small datasets

$$\Delta x_{ave} = \frac{\sigma}{\sqrt{N}} = \frac{x_{\max} - x_{\min}}{2\sqrt{N}}$$

Large datasets

$$\Delta x_{ave} = \frac{\sigma_N}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^N (x_i - x_{ave})^2}{(N-1)N}}$$

Multi-variable function

$$Y = f(x_1, x_2, \dots, x_N)$$

uncertainty ΔY depends on several experimental parameters x_1, x_2, \dots, x_N

The uncertainties of all variables have to be taken into account.

Error Propagation Function (EPF)

$$\Delta Y = \sqrt{\left(\frac{\partial f}{\partial x_1} \Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2} \Delta x_2\right)^2 + \dots}$$

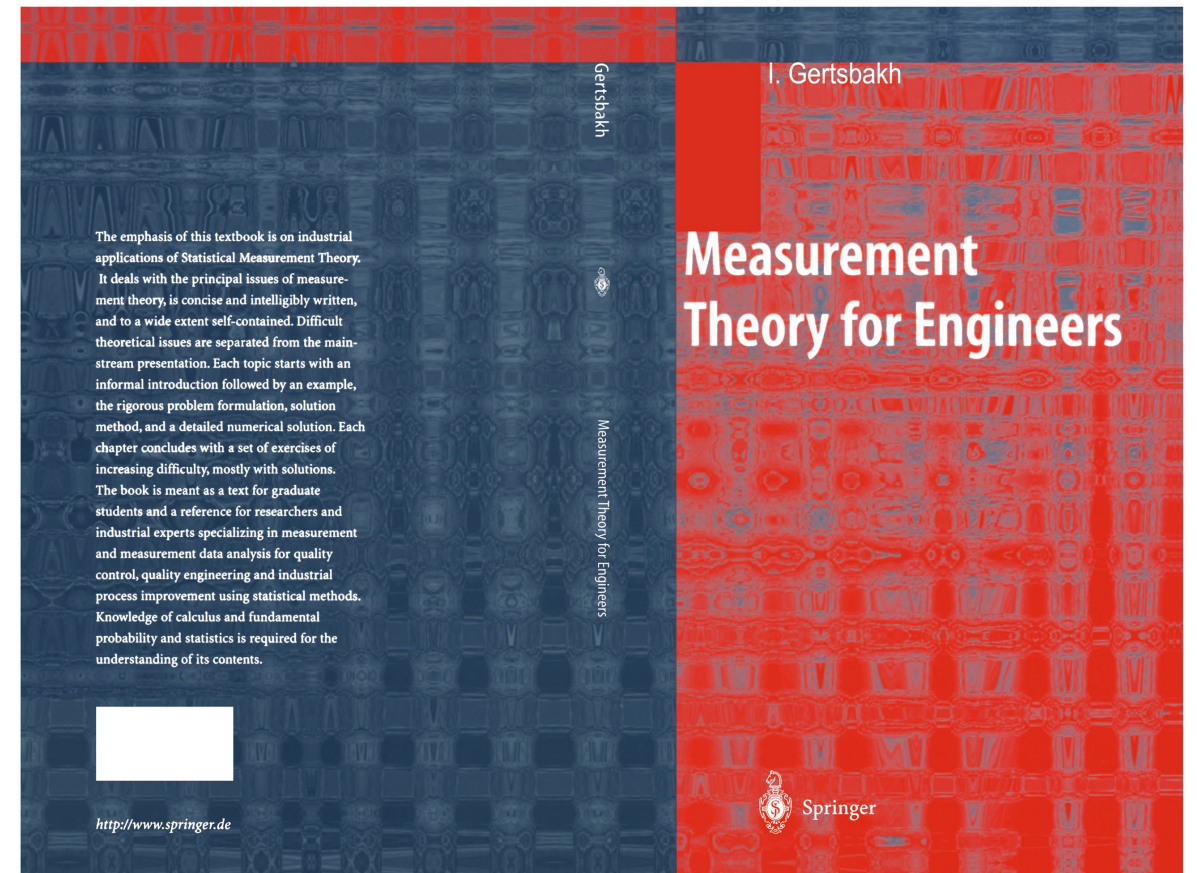
Further reading...

Chapters 2 : **You must know**

Chapters 3 : **You must know**

Chapters 4 : **Optional**

Chapters 5 : **You must know**



Quantum Measurement Theory

“Quantum Computation and Quantum Information”

Michael A. Nielsen & Isaac L. Chuang,

10th Anniversary Edition, §2.2.3 - §2.2.9



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The Hilbert space

In quantum mechanics, the state of the system is uniquely specified by a **vector in Hilbert space** $|q\rangle$. Hilbert space can have **finite** or infinite dimensions. Hilbert space vector components denote **probability amplitudes** related to the possible outcomes of certain measurements. This leads to the standard probabilistic interpretation of superpositions of Hilbert space vectors.

A linear combinations of two admissible vectors are again admissible vectors.

Dirac Notation

Ket

$$|q\rangle = \begin{pmatrix} q_1 \\ q_2 \\ \dots \\ q_d \end{pmatrix}$$

Bra

$$\langle q| = (q_1^* \quad q_2^* \quad \dots \quad q_d^*)$$

$$\langle |q\rangle \rangle^\dagger = \langle q|$$

Operators map states to each other linearly; they thus are $d \times d$ complex matrices operating on the d -dimensional Hilbert space:

$$Q|q\rangle \rightarrow |\psi\rangle$$

An **eigenstate** (or **eigenvector**) $|\psi_n\rangle$ of an operator Q fulfills the eigenvalue equation

$$Q|\psi_n\rangle \rightarrow c_n|\psi_n\rangle$$

where the complex number c_n is called the **eigenvalue**. The eigenvalues of different eigenstates can be equal; this is called **degeneracy**.



Basic maths in QM

$$|q\rangle\langle\psi| = \text{Outer product} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} (\psi_1 \quad \dots \quad \psi_n)$$

$n \times 1$ $1 \times n$ $n \times n$

$$\langle q|\psi\rangle = \text{Inner product} = (q_1 \quad \dots \quad q_n) \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

$1 \times n$ $n \times 1$ 1×1

$$\langle q|P|\psi\rangle = (q_1 \quad \dots \quad q_n) \begin{pmatrix} P_{11} & \dots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \dots & P_{nn} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

$1 \times n$ $n \times n$ $n \times 1$ 1×1

The Bloch sphere

A general expression for the qubit state is:

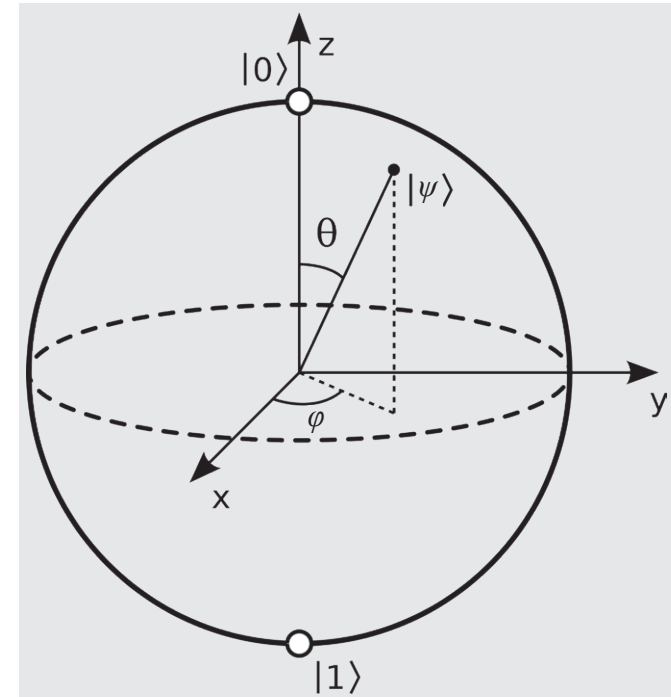
$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \quad |a_0|^2 + |a_1|^2 = 1$$

$$a_0 \rightarrow \cos(\theta/2) \quad a_1 \rightarrow \sin(\theta/2)$$

$$a_0 = e^{ia} \cos(\theta/2) \quad a_1 = e^{i(a+\phi)} \sin(\theta/2)$$

$$|\psi\rangle = e^{ia} \left(\cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle \right)$$

$$0 \leq \theta < \pi \quad 0 \leq \phi < 2\pi$$



$$e^{ia} = \cos(a) + i \sin(a)$$

The Bloch sphere

A general expression for the qubit state is:

$$|q\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

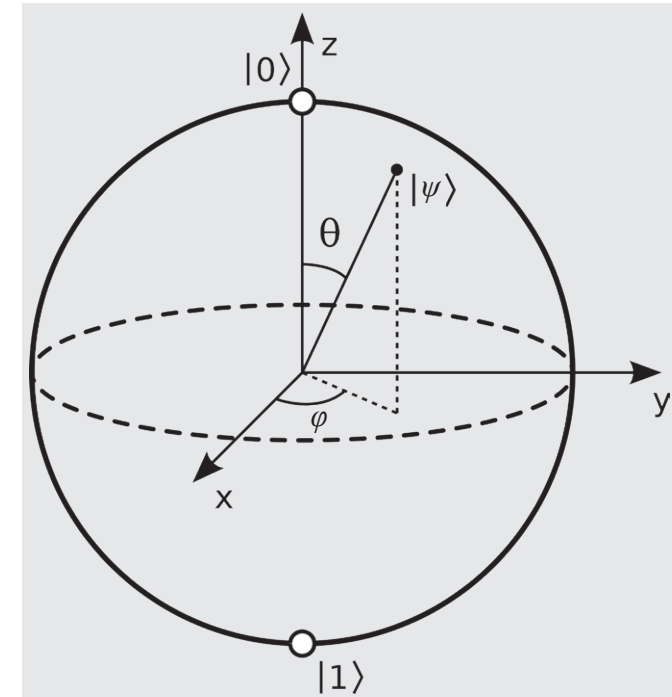
The state of the qubit is a vector in a Bloch sphere the radius of which equals to unity.

The qubit state vector initial point is the center of the sphere and the terminal point lies on its surface.

Angles ϑ and φ determine the direction of the state vector.

The two basis states are the poles of the sphere.

A qubit has an infinite number of superposition states



Basic QM measurement postulate

The state of a quantum system, $|\psi\rangle$, is a vector in a complex vector space. If the set of vectors $\{|n\rangle\}$, $n = 0, \dots, N - 1$ (where N may be ∞) is an orthonormal basis for this space, then we can always express $|\psi\rangle$ as

$$|\psi\rangle = \sum_n c_n |n\rangle \quad \text{for some complex coefficients } c_n, \text{ where } \sum_n |c_n|^2 = 1.$$

If a system is initially in the state $|\psi\rangle$ then the probability that we will find state $|n\rangle$ is given by $|c_n|^2$. Which basis state we find is random.

A measurement like this, for which the result is one of a set of basis states, is called a *von Neumann measurement*.

Measurement Techniques in Quantum Mechanics

- The measurement of a quantum system collapses the quantum system into an eigenstate of the operator corresponding to the measurement.
- The measurement of a quantum system produces a measurement result, the expected values of measurement are associated with a particular probability distribution.

In quantum mechanics several different measurement techniques exist.

In a **projection-valued measurement (PVM)**, the measurement of the quantum system is mathematically interpreted by projectors that project any initial quantum state onto one of the basis states. The projective measurement is also known as **von Neumann measurement**.

The projective measurement with no post-processing on the measurement results is referred to as **standard measurement** (It is motivated by the fact, that in a gate-model quantum computer environment the output quantum system is measured with respect to a particular computational basis).

Measurement Techniques in Quantum Mechanics

The **positive operator-valued measure (POVM)** is a generalized measurement that can be interpreted as a von Neumann measurement that utilizes an additional quantum system (called **ancilla**).

The POVM measurement is mathematically described by a set of positive operators such that their sum is the identity operator. The von Neumann measurements are a special case of a POVM measurement.

The POVM measurements therefore can be expressed in terms of projective measurements (Neumark's dilation theorem).

Another subject connected to quantum measurement theory is quantum-state discrimination that covers the distinguishability of quantum states, and the problem of differentiation between non-orthogonal quantum states.

In maths...

Observables (measurable quantities) correspond to **self-adjoint** or **Hermitian matrices**, that is,

$$\mathbf{A}^\dagger = \mathbf{A}; \quad (\mathbf{A}^\dagger)_{ij} := (\mathbf{A})_{ji}^*$$

Self-adjoint operators possess real eigenvalues (the eigenvalues are the possible outcomes of a measurement and thus have to be real); the eigenstates $|a_i\rangle$ corresponding to the eigenvalues a_i , of the operator \mathbf{A} are pairwise orthogonal (or can be orthogonalized in the case of degeneracy). Thus they form a basis in Hilbert space,

$$\mathbf{A}|a_i\rangle = a_i|a_i\rangle \quad \langle a_i|a_j\rangle = \delta_{ij} \quad (i, j = 1, \dots, d)$$

where δ_{ij} is the Kronecker symbol. (It should be kept in mind that we are operating in a finite-dimensional Hilbert space where all states can be normalized to unity.)

The sets of eigenstates and eigenvalues characterize an observable \mathbf{A} completely, because any arbitrary state can be expanded in eigenstates of \mathbf{A} which obey the above equation.

This leads to the **spectral representation** of \mathbf{A} .

In maths...

To define that representation we need a further class of operators: **projection operators** or **projectors** for short. The projector \mathbf{P}_i , onto the eigenstate $|a_i\rangle$ is defined by

$$\mathbf{P}_i := |a_i\rangle\langle a_i|$$

Application of \mathbf{P}_i to an arbitrary state $|\psi\rangle$ yields a multiple of $|a_i\rangle$

$$\mathbf{P}_i|\psi\rangle = |a_i\rangle\langle a_i|\psi\rangle = \langle a_i|\psi\rangle|a_i\rangle$$

where $|\langle a_i|\psi\rangle|$ the “length” of the projection of $|\psi\rangle$ onto the unit vector $|a_i\rangle$. And if $\langle a_i|a_j\rangle = \delta_{ij}$ then

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j; \text{ especially } \mathbf{P}_i^2 = \mathbf{P}_i$$

As the \mathbf{P}_i cover “all directions” of Hilbert space we obtain a completeness relation:

$$\sum_{i=1}^d \mathbf{P}_i = \sum_{i=1}^d |a_i\rangle\langle a_i| = \mathbf{1}$$

\mathbf{P}_i is Hermitian: $\mathbf{P}_i = \mathbf{P}_i^\dagger$

In maths...

The *spectral representation* of \mathbf{A} can be defined

$$\mathbf{A} = \sum_{i=1}^d a_i \mathbf{P}_i = \sum_{i=1}^d a_i |a_i\rangle \langle a_i|$$

Projection postulate

Assume a quantum system prepared in a state $|\psi\rangle$ and a single measurement of the observable \mathbf{A} is performed. This cycle of preparation and measurement is repeated many times so that the notion of probability used in the postulate makes sense.

Or imagine an ensemble containing a large number of independent copies of the quantum system, all prepared in the same state $|\psi\rangle$. \mathbf{A} is measured for all system copies independently.

Projection postulate : A single measurement of the observable \mathbf{A} in the normalized state $|\psi\rangle$ yields one of the eigenvalues a_i of \mathbf{A} with probability $|\langle a_i | \psi \rangle|^2$. Immediately after the measurement the system is in the (normalized) state

$$\frac{\mathbf{P}_i |\psi\rangle}{\|\mathbf{P}_i |\psi\rangle\|}$$

where \mathbf{P}_i is the projection operator onto the subspace of eigenstates of \mathbf{A} with eigenvalue a_i .

Projective measurement result

In general it is not possible to predict the outcome of a single measurement. A measurement of \mathbf{A} on an ensemble of systems as discussed above yields the *average* (expectation value)

$$\langle \mathbf{A} \rangle := \langle \psi | \mathbf{A} | \psi \rangle$$

with deviations described by the **variance** (the square of the standard deviation): $\langle (\mathbf{A} - \langle \mathbf{A} \rangle)^2 \rangle \geq 0$

The probability of obtaining outcome i for a given state $|\psi\rangle$

$$p_i = \langle \psi | \mathbf{P}_i | \psi \rangle$$

And the post-measurement state is given by

$$|\psi_i^{post}\rangle = \frac{\mathbf{P}_i |\psi\rangle}{\sqrt{\langle \psi | \mathbf{P}_i | \psi \rangle}}$$

In quantum mechanics the measurement change the state of a quantum system which is **probabilistic** and **irreversible** process. **The observation process is irreversible.**

End of Lesson