

⊗ LINEAR FUNCTIONALS

Theorem Let V be a finite-dimensional inner product space and f be a linear functional on V . Then there exists a unique vector $\underline{b} \in V$ such that

$$f(\underline{a}) = \langle \underline{a}, \underline{b} \rangle, \quad \forall \underline{a} \in V.$$

Proof Let $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ be an orthonormal basis for V .

We set

$$\underline{b} = \sum_{i=1}^n \overline{f(\underline{a}_i)} \cdot \underline{a}_i$$

Suppose, first, that $\underline{a} = \underline{a}_k$. Then

$$\langle \underline{a}_k, \underline{b} \rangle = \langle \underline{a}_k, \sum_{i=1}^n \overline{f(\underline{a}_i)} \underline{a}_i \rangle = \sum_{i=1}^n \langle \underline{a}_k, \overline{f(\underline{a}_i)} \underline{a}_i \rangle = \sum_{i=1}^n f(\underline{a}_i) \langle \underline{a}_k, \underline{a}_i \rangle = f(\underline{a}_k)$$

Suppose, now, that $\underline{a} \neq \underline{a}_k$. Then $\underline{a} = c_1 \underline{a}_1 + c_2 \underline{a}_2 + \dots + c_n \underline{a}_n = \sum_{i=1}^n c_i \underline{a}_i$

Thus

$$f(\underline{a}) = f\left(\sum_{i=1}^n c_i \underline{a}_i\right) = \sum_{i=1}^n c_i f(\underline{a}_i) = \sum_{i=1}^n c_i \langle \underline{a}_i, \underline{b} \rangle = \left\langle \sum_{i=1}^n c_i \underline{a}_i, \underline{b} \right\rangle = \langle \underline{a}, \underline{b} \rangle.$$

Finally, assume that there exists another vector $\underline{c}$ such that

$$f(\underline{a}) = \langle \underline{a}, \underline{c} \rangle, \quad \forall \underline{a} \in V.$$

But $f(\underline{a}) = \langle \underline{a}, \underline{b} \rangle, \quad \forall \underline{a} \in V$. Hence

$$\langle \underline{a}, \underline{c} \rangle = \langle \underline{a}, \underline{b} \rangle \Rightarrow \langle \underline{a}, \underline{c} \rangle - \langle \underline{a}, \underline{b} \rangle = 0 \Rightarrow \langle \underline{a}, \underline{c} - \underline{b} \rangle = 0, \quad \forall \underline{a} \in V.$$

So, for $\underline{a} = \underline{c} - \underline{b}$ we have

$$\langle \underline{c} - \underline{b}, \underline{c} - \underline{b} \rangle = 0 \Rightarrow \|\underline{c} - \underline{b}\|^2 = 0 \Rightarrow \|\underline{c} - \underline{b}\| = 0 \Rightarrow \underline{c} - \underline{b} = \underline{0} \Rightarrow \underline{c} = \underline{b}.$$

□

Remark Let V be a finite-dimensional space with the common inner product. Let $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ be an orthonormal basis for V .

We consider that f is any linear functional and $\underline{a}$ and $\underline{b}$ are two vectors of V such that

$$\underline{a} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n \quad \text{and} \quad \underline{b} = y_1 \underline{a}_1 + y_2 \underline{a}_2 + \dots + y_n \underline{a}_n. \quad \text{Then}$$

$$f(\underline{a}) = x_1 f(\underline{a}_1) + x_2 f(\underline{a}_2) + \dots + x_n f(\underline{a}_n). \quad \text{But } f(\underline{a}) = \langle \underline{a}, \underline{b} \rangle, \quad \text{i.e.}$$

$$f(\underline{a}) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \dots + x_n \bar{y}_n. \quad \text{Hence } \bar{y}_i = f(\underline{a}_i), \quad i = 1, 2, \dots, n$$

Therefore

$$y_i = \overline{f(\underline{a}_i)}, \quad i = 1, 2, \dots, n$$

and so

$$\underline{b} = \sum_{i=1}^n \overline{f(\underline{a}_i)} \cdot \underline{a}_i.$$

Remark Suppose that $f(\underline{a}) = \langle \underline{a}, \underline{b} \rangle = 0, \forall \underline{a} \in V$. Then

the vector $\underline{a}$ will belong to null space of f (i.e. W) and the vector $\underline{b}$ will belong to the orthogonal complement of W (i.e. $W^\perp$).

Therefore V will be the direct sum of W and $W^\perp$ (i.e. $V = W \oplus W^\perp$).

If $\underline{x} \in V$ and $\underline{x} = \underline{x}_1 + \underline{x}_2$ where $\underline{x}_1 \in W$ and $\underline{x}_2 \in W^\perp$, then

$$f(\underline{x}) = f(\underline{x}_1) + f(\underline{x}_2) = 0 + f(\underline{x}_2) = f(\underline{x}_2)$$

In this case f is determined only by the values on $W^\perp$.

If P is an orthogonal projection of V on $W^\perp$, then

$$\underline{a} = \underbrace{(I-P)\underline{a}}_{\in W} + \underbrace{P\underline{a}}_{\in W^\perp} \Rightarrow f(\underline{a}) = f(P\underline{a}), \forall \underline{a} \in V.$$

Thus

$$\dim V = \dim W + \dim W^\perp \Rightarrow$$

$$\dim V = \dim W + \operatorname{rank} f \Rightarrow$$

$$n = \dim W + 1 \Rightarrow$$

$$\dim W = n - 1.$$

Let $\underline{s} \in W^\perp$ and $\underline{s} \neq \underline{0}$. Then

$$P_{\underline{s}} = \frac{\langle \underline{a}, \underline{s} \rangle}{\|\underline{s}\|^2} \underline{s}.$$

Hence

$$f(\underline{a}) = f(P_{\underline{s}}) = \frac{\langle \underline{a}, \underline{s} \rangle}{\|\underline{s}\|^2} f(\underline{s}), \quad \forall \underline{a} \in V.$$

Moreover

$$\langle \underline{a}, \underline{b} \rangle = f(\underline{a}) = \frac{\langle \underline{a}, \underline{s} \rangle}{\|\underline{s}\|^2} f(\underline{s}) = \left\langle \underline{a}, \frac{\overline{f(\underline{s})}}{\|\underline{s}\|^2} \underline{s} \right\rangle.$$

Therefore

$$\underline{b} = \frac{\overline{f(\underline{s})}}{\|\underline{s}\|^2} \underline{s}.$$

Theorem Let V be a finite-dimensional inner product space. Let T be a linear operator, such that $T: V \rightarrow V$. Then there exists a unique linear operator T^* , such that

$$\langle T\underline{a}, \underline{b} \rangle = \langle \underline{a}, T^*\underline{b} \rangle, \quad \forall \underline{a}, \underline{b} \in V.$$

Proof Let $\underline{b} \in V$. Then, according to the last theorem, $\underline{a} \mapsto \langle T\underline{a}, \underline{b} \rangle$ is a linear functional on V . So $\exists \underline{b}' \in V$ such that $\langle T\underline{a}, \underline{b} \rangle = \langle \underline{a}, \underline{b}' \rangle, \forall \underline{a} \in V$. Consider the mapping T^* such that $\underline{b}' = T^*\underline{b}$. We must show that T^* is a linear operator. To this end, we consider $\underline{b}, \underline{x} \in V$ and $c_1, c_2 \in \mathbb{F}$. Then

$$\begin{aligned} \langle \underline{a}, T^*(c_1\underline{b} + c_2\underline{x}) \rangle &= \langle T\underline{a}, c_1\underline{b} + c_2\underline{x} \rangle = \langle T\underline{a}, c_1\underline{b} \rangle + \langle T\underline{a}, c_2\underline{x} \rangle = \\ &= \overline{c_1} \langle T\underline{a}, \underline{b} \rangle + \overline{c_2} \langle T\underline{a}, \underline{x} \rangle = \overline{c_1} \langle \underline{a}, T^*\underline{b} \rangle + \overline{c_2} \langle \underline{a}, T^*\underline{x} \rangle = \\ &= \langle \underline{a}, c_1 T^*\underline{b} + c_2 T^*\underline{x} \rangle = \langle \underline{a}, c_1 T^*\underline{b} + c_2 T^*\underline{x} \rangle. \end{aligned}$$

□

Example We consider the mapping

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \text{where} \quad T(\underline{a}) = T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ x_2 - x_3 \\ 2x_1 - x_2 + 2x_3 \end{pmatrix},$$

$$\underline{a} = (x_1, x_2, x_3) \quad \text{and} \quad \underline{b} = (y_1, y_2, y_3).$$

Then

$$\begin{aligned} \langle T\underline{a}, \underline{b} \rangle &= (x_1 - 2x_2 + x_3)y_1 + (x_2 - x_3)y_2 + (2x_1 - x_2 + 2x_3)y_3 \\ &= x_1(y_1 + 2y_3) + x_2(-2y_1 + y_2 - y_3) + x_3(y_1 - y_2 + 2y_3) \\ &= \langle \underline{a}, T^*\underline{b} \rangle \end{aligned}$$

Therefore

$$T^* \underline{b} = \begin{pmatrix} y_1 & +2y_3 \\ -2y_1 + y_2 & -y_3 \\ y_1 - y_2 + 2y_3 \end{pmatrix}.$$

