

⊗ ORTHOGONAL PROJECTION

Definition Let W be a subspace of an inner product space V and $b \in V$. Suppose that $\exists \underline{a} \in W$, such that

$$\langle b - \underline{a}, \underline{c} \rangle = 0, \quad \forall \underline{c} \in W.$$

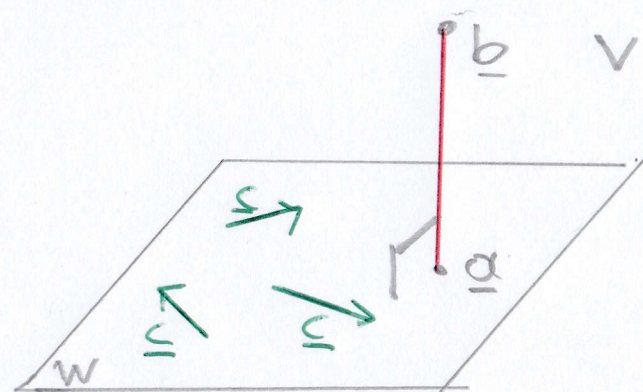
Then the vector $\underline{a}$ is called the orthogonal projection of $\underline{b}$ on W .

If every vector in V has an orthogonal projection on W ,

then the mapping E that corresponds every vector of V to

its orthogonal projection, is called

orthogonal projection E of V on W .



Theorem Let W be a finite-dimensional subspace of an inner product space V and let $\underline{b} \in V$. Suppose that E is the orthogonal projection of V on W . Then the mapping $I - E$ (i.e. $\underline{b} \xrightarrow{I-E} \underline{b} - E\underline{b}$) is the orthogonal projection of V on $W^\perp$, where I is the identity mapping.

Proof Let $\underline{c}$ be an arbitrary vector of $W^\perp$.

It is sufficient to show that

$$\|\underline{b} - (\underline{b} - E\underline{b})\| \leq \|\underline{b} - \underline{c}\|, \quad \forall \underline{c} \in W^\perp.$$

It holds that,

$$\begin{aligned}\|\underline{b} - \underline{c}\|^2 &= \|\underline{E}\underline{b} - \underline{E}\underline{b} + \underline{b} - \underline{c}\|^2 = \|\underline{E}\underline{b} + (\underline{b} - \underline{E}\underline{b} - \underline{c})\|^2 \\ &= \|\underline{E}\underline{b}\|^2 + 2\operatorname{Re}\langle \underline{E}\underline{b}, \underline{b} - \underline{E}\underline{b} - \underline{c} \rangle + \|\underline{b} - \underline{E}\underline{b} - \underline{c}\|^2\end{aligned}$$

But $\underline{b} - \underline{E}\underline{b} \in W^\perp$ and $\underline{c} \in W^\perp$, so $\underline{b} - \underline{E}\underline{b} - \underline{c} \in W^\perp$.

Hence $\underline{E}\underline{b} \perp \underline{b} - \underline{E}\underline{b} - \underline{c}$ and so $\langle \underline{E}\underline{b}, \underline{b} - \underline{E}\underline{b} - \underline{c} \rangle = 0$.

$$\text{Thus } \|\underline{b} - \underline{c}\|^2 = \|\underline{E}\underline{b}\|^2 + \|\underline{b} - \underline{E}\underline{b} - \underline{c}\|^2 \geq \|\underline{E}\underline{b}\|^2 = \|\underline{b} + \underline{E}\underline{b} - \underline{b}\|^2 \Rightarrow$$

$$\|\underline{b} - \underline{c}\|^2 \geq \|\underline{b} - (\underline{b} - \underline{E}\underline{b})\|^2 \Rightarrow \|\underline{b} - (\underline{b} - \underline{E}\underline{b})\|^2 \leq \|\underline{b} - \underline{c}\|^2 \Rightarrow$$

$$\|\underline{b} - (\underline{b} - \underline{E}\underline{b})\| \leq \|\underline{b} - \underline{c}\|, \quad \forall \underline{c} \in W^\perp$$

i.e. the vector $\underline{b} - \underline{E}\underline{b}$ is the orthogonal projection of V on $W^\perp$.

□

Theorem Let W be a finite-dimensional subspace of an inner product space V . Let E be the orthogonal projection of V on W .

Then the following statements are true:

- (i) $E^2 = E$ (i.e. E is an idempotent mapping).
- (ii) E is a linear mapping
- (iii) The subspace $W^\perp$ is null space (kernel) of E .
- (iv) $V = W \oplus W^\perp$.

Proof (i) Let $\underline{c} \in W$. Then $E\underline{c} = \underline{c}$. Thus $E(E\underline{c}) = E\underline{c}$ and so

$$E^2 \underline{c} = E\underline{c}.$$

(ii) Let $\underline{a}_1, \underline{a}_2 \in V$ and $\underline{c}_1, \underline{c}_2 \in F$ where F is a field. Then it is sufficient to show that

$$E(\underline{c}_1 \underline{a}_1 + \underline{c}_2 \underline{a}_2) = \underline{c}_1 E\underline{a}_1 + \underline{c}_2 E\underline{a}_2$$

or equivalently $\langle c_1 \underline{a}_1 + c_2 \underline{a}_2 - (c_1 E \underline{a}_1 + c_2 E \underline{a}_2), \underline{c} \rangle = 0, \forall \underline{c} \in W.$

Indeed

$$\begin{aligned} c_1 \underline{a}_1 + c_2 \underline{a}_2 - (c_1 E \underline{a}_1 + c_2 E \underline{a}_2) &= c_1 \underline{a}_1 + c_2 \underline{a}_2 - c_1 E \underline{a}_1 - c_2 E \underline{a}_2 \\ &= c_1 \underline{a}_1 - c_1 E \underline{a}_1 + c_2 \underline{a}_2 - c_2 E \underline{a}_2 = c_1 (\underline{a}_1 - E \underline{a}_1) + c_2 (\underline{a}_2 - E \underline{a}_2). \end{aligned}$$

But $\underline{a}_1 - E \underline{a}_1 \in W^\perp$ and $\underline{a}_2 - E \underline{a}_2 \in W^\perp$. Hence $c_1 (\underline{a}_1 - E \underline{a}_1) + c_2 (\underline{a}_2 - E \underline{a}_2) \in W^\perp$.

(iii) Let $\underline{b} \in W^\perp$. Then $\langle \underline{b}, \underline{c} \rangle = 0, \forall \underline{c} \in W$. Furthermore $\langle \underline{b} - E \underline{b}, \underline{c} \rangle = 0, \forall \underline{c} \in W$. Since the best approximation is unique, we get $\underline{b} = \underline{b} - E \underline{b} \Rightarrow E \underline{b} = \underline{0}$.

Therefore the subspace $W^\perp$ is the null space of E .

(iv) Let $\underline{a} \in V$. Then $\underline{a} = E \underline{a} + \underbrace{\underline{a} - E \underline{a}}_{\in W^\perp} = \underbrace{E \underline{a}}_{\in W} + \underbrace{(\underline{a} - E \underline{a})}_{\in W^\perp} \Rightarrow V = W + W^\perp$.

Suppose that $\exists x \in W \cap W^\perp$ and $x \neq \underline{0}$.

Since $x \in W$ and $x \in W^\perp$, $\langle x, x \rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow \|x\| = 0 \Rightarrow x = \underline{0}$.

which is a contradiction. Therefore the $W \cap W^\perp = \{\underline{0}\}$.

□

Theorem Let W be a finite-dimensional subspace of an inner product space V . Let $I-E$ be the orthogonal projection of V on $W^\perp$. Then the following statements are true.

(i) $(I-E)^2 = I-E$ (i.e. $I-E$ is an Idempotent mapping).

(ii) $I-E$ is a linear mapping.

(ii') The subspace W is the null space of $I-E$.

Proof. (i)
$$\begin{aligned} (I-E)^2 &= (I-E)(I-E) \\ &= I^2 - E - E + E^2 \\ &= I - E - \cancel{E} + \cancel{E} \\ &= I - E \end{aligned}$$

(ii) We know that I is a linear mapping.
We proved that E is a linear mapping.
Hence, $I-E$ will be a linear mapping, too.

$$\begin{aligned} \text{(iii)} \quad (I-E)\underline{b} = \underline{0} &\Rightarrow \underline{b} - E\underline{b} = \underline{0} \\ &\Rightarrow E\underline{b} = \underline{b} \\ &\Rightarrow \underline{b} \in W \end{aligned}$$

Therefore W is the null space of $I-E$.



Example Let $V = \mathbb{R}^3$ and W be a subspace of V that is spanned by the vector $\underline{a} = (3, 12, -1)$.

- Find the orthogonal projection E of V on W .
- Find the orthogonal complement $W^\perp$.
- Find the dimension of W .

Solution a) Let $\underline{x} = (x_1, x_2, x_3)$ be an arbitrary vector of $V = \mathbb{R}^3$.

Then the orthogonal projection of $\underline{x}$ on W is

$$\begin{aligned} E\underline{x} &= \frac{\langle \underline{x}, \underline{a} \rangle}{\|\underline{a}\|^2} \underline{a} = \frac{3x_1 + 12x_2 - x_3}{3^2 + 12^2 + (-1)^2} (3, 12, -1) = \frac{3x_1 + 12x_2 - x_3}{154} (3, 12, -1) \\ &= \left(\frac{9x_1 + 36x_2 - 3x_3}{154}, \frac{36x_1 + 144x_2 - 12x_3}{154}, \frac{-3x_1 - 12x_2 + x_3}{154} \right) \end{aligned}$$

$$(b) E_{\underline{x}} = \underline{0} \Rightarrow \frac{1}{154} (9x_1 + 36x_2 - 3x_3, 36x_1 + 144x_2 - 12x_3, -3x_1 - 12x_2 + x_3) = \underline{0}$$

$$\left. \begin{aligned} 9x_1 + 36x_2 - 3x_3 &= 0 \\ \Rightarrow 36x_1 + 144x_2 - 12x_3 &= 0 \\ -3x_1 + 12x_2 + x_3 &= 0 \end{aligned} \right\} \Rightarrow -3x_1 - 12x_2 + x_3 = 0$$

Thus

$$\begin{aligned} W^\perp &= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid -3x_1 - 12x_2 + x_3 = 0 \} \\ &= \{ (x_1, x_2, 3x_1 + 12x_2) \mid x_1, x_2 \in \mathbb{R} \} \\ &= \{ x_1(1, 0, 3) + x_2(0, 1, 12) \mid x_1, x_2 \in \mathbb{R} \} \end{aligned}$$

(c) We obtain $\dim W^\perp = 2$. Therefore

$$\begin{aligned} \dim \mathbb{R}^3 &= \dim W + \dim W^\perp \Rightarrow \\ 3 &= \dim W + 2 \Rightarrow \\ \dim W &= 1. \end{aligned}$$

□

Alternatively, we could find the space W and the dimension of W .

To this end, we consider an arbitrary vector $\underline{y} = (y_1, y_2, y_3)$ of $V = \mathbb{R}^3$.

Then

$$(\underline{I} - \underline{E})\underline{y} = \underline{0} \Rightarrow \underline{I}\underline{y} - \underline{E}\underline{y} = \underline{0} \Rightarrow \underline{y} - \underline{E}\underline{y} = \underline{0} \Rightarrow$$

$$\left(y_1 - \frac{9y_1 + 36y_2 - 3y_3}{154}, \quad y_2 - \frac{36y_1 + 144y_2 - 12y_3}{154}, \quad y_3 - \frac{-3y_1 - 12y_2 + y_3}{154} \right) = (0, 0, 0) \Rightarrow$$

$$\frac{1}{154} (145y_1 - 36y_2 + 3y_3, \quad -36y_1 + 10y_2 + 12y_3, \quad 3y_1 + 12y_2 + 153y_3) = (0, 0, 0) \Rightarrow$$

$$\left. \begin{aligned} 145y_1 - 36y_2 + 3y_3 &= 0 \\ -36y_1 + 10y_2 + 12y_3 &= 0 \\ 3y_1 + 12y_2 + 153y_3 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} 145y_1 - 36y_2 + 3y_3 &= 0 \\ -36y_1 + 10y_2 + 12y_3 &= 0 \\ y_1 + 4y_2 + 51y_3 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} 145y_1 - 36y_2 + 3y_3 &= 0 \\ -36y_1 + 10y_2 + 12y_3 &= 0 \\ y_1 = -4y_2 - 51y_3 &= 0 \end{aligned} \right\} \Rightarrow$$

$$\begin{array}{l}
 145(-4y_2 - 51y_3) - 36y_2 + 3y_3 = 0 \\
 -18(-4y_2 - 51y_3) + 5y_2 + 6y_3 = 0 \\
 y_1 = -4y_2 - 51y_3
 \end{array}
 \Rightarrow
 \begin{array}{l}
 -580y_2 - 7395y_3 - 36y_2 + 3y_3 = 0 \\
 72y_2 + 918y_3 + 5y_2 + 6y_3 = 0 \\
 y_1 = -4y_2 - 51y_3
 \end{array}
 \Rightarrow$$

$$\begin{array}{l}
 -616y_2 - 7392y_3 = 0 \\
 77y_2 + 924y_3 = 0 \\
 y_1 = -4y_2 - 51y_3
 \end{array}
 \begin{array}{l}
 :8 \\
 :77
 \end{array}
 \Rightarrow
 \begin{array}{l}
 77y_2 + 924y_3 = 0 \\
 77y_2 + 924y_3 = 0 \\
 y_1 = -4y_2 - 51y_3
 \end{array}
 \Rightarrow
 \begin{array}{l}
 y_2 + 12y_3 = 0 \\
 y_1 = -4y_2 - 51y_3
 \end{array}
 \Rightarrow$$

$$\Rightarrow y_1 = -4(-12y_3) - 51y_3 = -3y_3, \quad y_2 = -12y_3, \quad y_3 \in \mathbb{R}. \text{ Therefore}$$

$$\begin{aligned}
 W &= \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 = -3y_3, y_2 = -12y_3\} = \{(-3y_3, -12y_3, y_3) \in \mathbb{R}^3 \mid y_3 \in \mathbb{R}\} \\
 &= \{c(-3, -12, 1) \mid c \in \mathbb{R}\},
 \end{aligned}$$

Hence $\dim W = 1$
and so $\dim W^\perp = 3 - 1 = 2$.

