

⊙ BEST APPROXIMATION

Def. Let W be a subspace of an inner product space V and $\underline{b} \in V$. Then best approximation of $\underline{b}$ by vectors in W is called a vector $\underline{a}$ in W such that

$$\|\underline{b} - \underline{a}\| \leq \|\underline{b} - \underline{c}\|, \quad \forall \underline{c} \in W.$$

Theorem. Let V be inner product space and $W \subset V$. Let $\underline{b} \in V$.

Then the following statements are true:

- (i) The vector $\underline{a}$ in W is a best approximation to $\underline{b}$ by vector in W iff $\underline{b} - \underline{a}$ is orthogonal to every vector in W .
- (ii) If there exists a best approximation to $\underline{b}$ by vectors in W then it is unique.
- (iii) If W is a finite-dimensional vector space and $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ is any orthogonal basis for W , then the vector $\underline{a} = \sum_{k=1}^n \frac{\langle \underline{b}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \underline{a}_k$ is the unique best approximation to $\underline{b}$ by vectors in W .

Proof: (1) " $\Leftarrow$ " First suppose that $\underline{b} - \underline{a}$ is orthogonal to every vector in W . Let $\underline{c}$ be a vector in W such that $\underline{c} \neq \underline{a}$. Then $\underline{b} - \underline{c} = \underline{b} - \underline{a} + \underline{a} - \underline{c}$.

But $\underline{a} - \underline{c} \in W$, so

$$\begin{aligned}\|\underline{b} - \underline{c}\|^2 &= \|(\underline{b} - \underline{a}) + (\underline{a} - \underline{c})\|^2 = \|\underline{b} - \underline{a}\|^2 + 2\operatorname{Re}\langle \underline{b} - \underline{a}, \underline{a} - \underline{c} \rangle + \|\underline{a} - \underline{c}\|^2 \\ &= \|\underline{b} - \underline{a}\|^2 + \|\underline{a} - \underline{c}\|^2 \Rightarrow \|\underline{b} - \underline{c}\|^2 \geq \|\underline{b} - \underline{a}\|^2 \Rightarrow \|\underline{b} - \underline{a}\| \leq \|\underline{b} - \underline{c}\|\end{aligned}$$

" $\Rightarrow$ " Now, suppose that $\underline{a} \in W$ is a best approximation to $\underline{b} \in V$ by vectors in W . Then, it holds that, $\forall \underline{c} \in W$, $\|\underline{b} - \underline{a}\| \leq \|\underline{b} - \underline{c}\|$. We proved in (i) that

$$\begin{aligned}2\operatorname{Re}\langle \underline{b} - \underline{a}, \underline{a} - \underline{c} \rangle + \|\underline{a} - \underline{c}\|^2 &= \|(\underline{b} - \underline{a}) + (\underline{a} - \underline{c})\|^2 - \|\underline{b} - \underline{a}\|^2 \\ &= \|\underline{b} - \underline{c}\|^2 - \|\underline{b} - \underline{a}\|^2 \geq 0.\end{aligned}$$

If $\underline{z}$ is an arbitrary vector in W , then

$$2\operatorname{Re}\langle \underline{b} - \underline{a}, \underline{z} \rangle + \|\underline{z}\|^2 \geq 0.$$

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We set $\underline{z} = -\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}(\underline{a}-\underline{c})$. Then

$$2\operatorname{Re}\langle \underline{b}-\underline{a}, -\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}(\underline{a}-\underline{c}) \rangle + \left\| -\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}(\underline{a}-\underline{c}) \right\|^2 \geq 0 \Rightarrow$$

$$2\operatorname{Re}\left(-\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}\right)\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle + \left\langle -\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}(\underline{a}-\underline{c}), \frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}(\underline{a}-\underline{c}) \right\rangle \geq 0 \Rightarrow$$

$$2\operatorname{Re}\left(-\frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2}\right)\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle + \frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}{\|\underline{a}-\underline{c}\|^2} \cdot \frac{\overline{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle}}{\|\underline{a}-\underline{c}\|^2} \langle \underline{a}-\underline{c}, \underline{a}-\underline{c} \rangle \geq 0 \Rightarrow$$

$$-2 \frac{\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle^2}{\|\underline{a}-\underline{c}\|^2} + \frac{|\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle|^2}{\|\underline{a}-\underline{c}\|^2} \geq 0 \Rightarrow -\frac{|\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle|^2}{\|\underline{a}-\underline{c}\|^2} \geq 0 \Rightarrow$$

$$\frac{|\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle|^2}{\|\underline{a}-\underline{c}\|^2} \leq 0 \Rightarrow |\langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle|^2 = 0 \Rightarrow \langle \underline{b}-\underline{a}, \underline{a}-\underline{c} \rangle = 0 \Rightarrow$$

$$(\underline{b}-\underline{a}) \perp (\underline{a}-\underline{c}) \Rightarrow \underline{b}-\underline{a} \perp \underline{W}, \quad \forall \underline{w} \in \underline{W}.$$

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(ii) Assume that $\exists \underline{a}, \underline{a}' \in W$ such that $\underline{a}, \underline{a}'$ are both best approximations to $\underline{b} \in V$ by vectors in W . Then $\underline{a} - \underline{a}' \in W$ and according to (i) it holds

$$\langle \underline{b} - \underline{a}, \underline{a} - \underline{a}' \rangle = 0 \text{ and } \langle \underline{b} - \underline{a}', \underline{a} - \underline{a}' \rangle = 0.$$

Hence

$$\begin{aligned} \|\underline{a} - \underline{a}'\|^2 &= \langle \underline{a} - \underline{a}', \underline{a} - \underline{a}' \rangle = \langle \underline{a} - \underline{b} + \underline{b} - \underline{a}', \underline{a} - \underline{a}' \rangle \\ &= \langle \underline{a} - \underline{b}, \underline{a} - \underline{a}' \rangle + \langle \underline{b} - \underline{a}', \underline{a} - \underline{a}' \rangle = 0 + 0 = 0. \Rightarrow \end{aligned}$$

$$\|\underline{a} - \underline{a}'\|^2 = 0 \Rightarrow \|\underline{a} - \underline{a}'\| = 0 \Rightarrow \underline{a} - \underline{a}' = \underline{0} \Rightarrow \underline{a} = \underline{a}'.$$

(iii) Applying the Gram-Schmidt orthogonalization process, we obtain

$$\underline{a}_{m+1} = \underline{b} - \sum_{k=1}^n \frac{\langle \underline{b}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \underline{a}_k = \underline{b} - \underline{a}. \text{ Then}$$

$$\langle \underline{a}_{m+1}, \underline{a}_1 \rangle = \langle \underline{a}_{m+1}, \underline{a}_2 \rangle = \dots = \langle \underline{a}_{m+1}, \underline{a}_m \rangle = 0. \text{ Thus}$$

$$\langle \underline{b} - \underline{a}, \underline{a}_1 \rangle = \langle \underline{b} - \underline{a}, \underline{a}_2 \rangle = \dots = \langle \underline{b} - \underline{a}, \underline{a}_m \rangle = 0 \text{ i.e. } \underline{b} - \underline{a} \perp W, \forall \underline{w} \in W.$$

Therefore, by (i), we get that $\underline{a}$ is the best approximation to $\underline{b}$ and according to (ii) it is unique. □

Definition Let V be an inner product space and let $S \subset V$. Then orthogonal complement of S is called the set $S^\perp$ of all vectors in V which are orthogonal to every vector in S .

Remark The orthogonal complement of V is $\{0\}$ and conversely.

Proposition Let V be an inner product space and let $S \subset V$. Then the set $S^\perp$ is a subspace of V .

Proof: Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and let $\underline{a}_1, \underline{a}_2 \in S^\perp$. Moreover let $\underline{c} \in S$. Then

$$\langle \lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2, \underline{c} \rangle = \langle \lambda_1 \underline{a}_1, \underline{c} \rangle + \langle \lambda_2 \underline{a}_2, \underline{c} \rangle = \lambda_1 \langle \cancel{\underline{a}_1}, \overset{0}{\underline{c}} \rangle + \lambda_2 \langle \cancel{\underline{a}_2}, \overset{0}{\underline{c}} \rangle = 0 + 0 = 0.$$

Hence the vector $\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2$ is in $S^\perp$ and so $S^\perp$ is a subspace of V .

□