

LECTURE 9: | ORTHOGONAL VECTORS

Definition

Consider a vector space V and let $\underline{a}, \underline{b} \in V$.

Then $\underline{a}, \underline{b}$ are called **orthogonal vectors** iff

$$\langle \underline{a}, \underline{b} \rangle = 0$$

Definition

Consider a vector space V and let S be a subset of V such that $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ where $\underline{a}_i \in V$, for $i = 1, 2, \dots, n$.

Then S is called **orthogonal set** iff

$$\langle \underline{a}_i, \underline{a}_j \rangle = 0, \quad \forall \underline{a}_i, \underline{a}_j \in S.$$

Furthermore S is called **orthonormal set** iff it is an orthogonal set and

$$\|\underline{a}_i\| = 1, \quad \forall \underline{a}_i \in S.$$

Theorem

Consider the vector space V and let S be a subset of V such that S is an orthogonal set of non zero vectors. Then S is a linearly independent set.

Proof

Let $S = \{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$. Then we have

$$\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \dots + \lambda_n \underline{a}_n = \underline{0} \Rightarrow$$

$$\lambda_1 \langle \underline{a}_1, \underline{a}_k \rangle + \lambda_2 \langle \underline{a}_2, \underline{a}_k \rangle + \dots + \lambda_n \langle \underline{a}_n, \underline{a}_k \rangle = \underline{0} \Rightarrow$$

$$0 + \dots + \lambda_k \langle \underline{a}_k, \underline{a}_k \rangle + \dots + 0 = \underline{0} \Rightarrow$$

$$\lambda_k \|\underline{a}_k\|^2 = 0, \quad \forall k \in \{1, 2, \dots, n\}$$

Therefore $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$, i.e. the set S is linearly independent. ■

Example

Consider the Euclidean vector space $\mathbb{R}^2$ with the inner product

$$\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^2 a_i b_i$$

where $\underline{a} = (a_1, a_2) \in \mathbb{R}^2$ and $\underline{b} = (b_1, b_2) \in \mathbb{R}^2$.

Then every vector

$$\underline{x} = (x_1, x_2) \in \mathbb{R}^2$$

will be orthogonal to the vector

$$\underline{x}' = (-x_2, x_1) \in \mathbb{R}^2.$$

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Example

Consider the Euclidean vector space $C[0,1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$$

where

$$\begin{aligned} f_n(t) &= \sqrt{2} \cos(2n\pi t), \\ g_n(t) &= \sqrt{2} \sin(2n\pi t). \end{aligned}$$

Show that the set

$$\{1, f_1, g_1, f_2, g_2, \dots\}$$

Is an orthonormal set of vectors.

Solution

It holds that

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

$$2 \cos a \cos b = \cos(a + b) + \cos(a - b)$$

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

$$\sin(a + b) = \sin a \cos b + \sin b \cos a$$

$$\sin(a - b) = \sin a \cos b - \sin b \cos a$$

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b)$$

$$2 \sin b \cos a = \sin(a + b) - \sin(a - b)$$

$$\cos^2 a = \frac{1 + \cos 2a}{2}, \quad \sin^2 a = \frac{1 - \cos 2a}{2}$$

$$\sin 2a = 2 \sin a \cos a$$

$$\langle 1, f_n \rangle = \int_0^1 1 \cdot \sqrt{2} \cos(2\pi nx) dx = \frac{\sqrt{2}}{2n\pi} [\sin(2\pi nx)]_{x=0}^1 dx = 0.$$

$$\langle 1, g_n \rangle = \int_0^1 1 \cdot \sqrt{2} \sin(2\pi nx) dx = \frac{\sqrt{2}}{2n\pi} [-\cos(2\pi nx)]_{x=0}^1 dx = 0.$$

$$\begin{aligned} \langle f_n, g_n \rangle &= \int_0^1 \sqrt{2} \cos(2\pi nx) \cdot \sqrt{2} \sin(2\pi nx) dx \\ &= \cancel{2} \int_0^1 \frac{1}{\cancel{2}} \sin 2(2\pi nx) dx = \frac{1}{4n\pi} [-\cos(4\pi nx)]_{x=0}^1 dx = 0. \end{aligned}$$

$$\begin{aligned} \langle f_n, g_m \rangle &= \int_0^1 \sqrt{2} \cos(2\pi nx) \cdot \sqrt{2} \sin(2\pi mx) dx = \int_0^1 [\sin 2(n+m)\pi x - \sin 2(m-n)\pi x] dx \\ &= \frac{1}{2(n+m)\pi} [-\cos 2(n+m)\pi x]_{x=0}^1 - \frac{1}{2(m-n)\pi} [-\cos 2(m-n)\pi x]_{x=0}^1 = 0. \end{aligned}$$

$$\begin{aligned}\langle f_n, f_m \rangle &= \int_0^1 \sqrt{2} \cos(2\pi nx) \cdot \sqrt{2} \cos(2\pi mx) dx \\ &= \int_0^1 [\cos 2(n+m)\pi x + \cos 2(n-m)\pi x] dx \\ &= \frac{1}{2(n+m)\pi} [\sin 2(n+m)\pi x]_{x=0}^1 + \frac{1}{2(n-m)\pi} [\sin 2(n-m)\pi x]_{x=0}^1 = 0, \text{ for } n \neq m.\end{aligned}$$

$$\begin{aligned}\langle g_n, g_m \rangle &= \int_0^1 \sqrt{2} \sin(2\pi nx) \cdot \sqrt{2} \sin(2\pi mx) dx \\ &= \int_0^1 [\cos 2(n+m)\pi x - \cos 2(n-m)\pi x] dx \\ &= \frac{1}{2(n+m)\pi} [\sin 2(n+m)\pi x]_{x=0}^1 - \frac{1}{2(n-m)\pi} [\sin 2(n-m)\pi x]_{x=0}^1 = 0, \text{ for } n \neq m.\end{aligned}$$

$$\langle 1, 1 \rangle = \int_0^1 1 \cdot 1 dx = 1$$

$$\begin{aligned} \langle f_n, f_{nm} \rangle &= \int_0^1 \sqrt{2} \cos(2\pi nx) \cdot \sqrt{2} \cos(2\pi nx) dx = 2 \int_0^1 \cos^2(2\pi nx) dx \\ &= \cancel{2} \frac{1}{\cancel{2}} \int_0^1 (1 + \cos 4n\pi x) dx = \left[x + \frac{1}{4n\pi} \sin 4n\pi x \right]_{x=0}^1 = 1. \end{aligned}$$

$$\begin{aligned} \langle g_n, g_{nm} \rangle &= \int_0^1 \sqrt{2} \sin(2\pi mx) \cdot \sqrt{2} \sin(2\pi mx) dx = 2 \int_0^1 \sin^2(2\pi mx) dx \\ &= \cancel{2} \frac{1}{\cancel{2}} \int_0^1 (1 - \cos 4n\pi x) dx = \left[x - \frac{1}{4n\pi} \sin 4n\pi x \right]_{x=0}^1 = 1. \end{aligned}$$

Therefore S is an orthonormal set of vectors.

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