

LECTURE 7:

LINEAR INDEPENDENCE AND BASE

Definition

Consider a vector space V and let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in V$.

Then these vectors are called **linearly dependent**, if $\exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ which are not all equal to zero, such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \dots + \lambda_n \underline{x}_n = \underline{0}.$$

Otherwise these vectors are called **linearly independent**.

Definition

Consider a vector space V and let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in V$.

Then we say that these vectors **span** V , if every vector $\underline{x} \in V$ can be written as

$$\underline{x} = \lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \dots + \lambda_n \underline{x}_n.$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$.

Definition

Consider a vector space V and let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in V$.

Then these vectors are called **base** of V , if these vectors are linearly independent and span V .

Theorem

Consider a vector space V and let $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n \in V$.

Then the set of vectors is a **base** of V , if and only if every vector of V can be written in a unique way as a linear combination of these vectors.

Definition

Consider a vector space V .

Then **dimension** of V is called the number of vectors of any base of V .

Example

Show that the vectors

$$\underline{x}_1 = (6, 3, 2, 4), \quad \underline{x}_2 = (0, 5, -3, 1), \quad \underline{x}_3 = (0, 0, 7, -2),$$

of $\mathbb{R}^4$ are linearly independent vectors of $\mathbb{R}^4$.

Solution

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3 = \underline{0} \Rightarrow \lambda_1 (6, 3, 2, 4) + \lambda_2 (0, 5, -3, 1) + \lambda_3 (0, 0, 7, -2) = \underline{0} \Rightarrow$$

$$(6\lambda_1, \quad 2\lambda_1 + 5\lambda_2, \quad 3\lambda_1 - 3\lambda_2 + 7\lambda_3, \quad 4\lambda_1 + \lambda_2 - 2\lambda_3) = (0, 0, 0, 0) \Rightarrow$$

$$\left. \begin{array}{lcl} 6\lambda_1 & = & 0 \\ 2\lambda_1 + 5\lambda_2 & = & 0 \\ 3\lambda_1 - 3\lambda_2 + 7\lambda_3 & = & 0 \\ 4\lambda_1 + \lambda_2 - 2\lambda_3 & = & 0 \end{array} \right\} \Rightarrow \begin{array}{lcl} \lambda_1 & = & 0 \\ \lambda_2 & = & 0 \\ \lambda_3 & = & 0 \end{array}$$

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Example

Show that the vectors

$$\underline{x}_1 = (1, -1, 0), \quad \underline{x}_2 = (1, 3, -1), \quad \underline{x}_3 = (5, 3, -2),$$

of $\mathbb{R}^3$ are linearly dependent vectors of $\mathbb{R}^3$.

Solution

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3 = \underline{0} \Rightarrow \lambda_1 (1, -1, 0) + \lambda_2 (1, 3, -1) + \lambda_3 (5, 3, -2) = \underline{0} \Rightarrow$$

$$(\lambda_1 + \lambda_2 + 5\lambda_3, \quad -\lambda_1 + 3\lambda_2 + 3\lambda_3, \quad -\lambda_2 - 2\lambda_3) = (0, 0, 0) \Rightarrow$$

$$\left. \begin{array}{rcl} \lambda_1 + \lambda_2 + 5\lambda_3 & = & 0 \\ -\lambda_1 + 3\lambda_2 + 3\lambda_3 & = & 0 \\ -\lambda_2 - 2\lambda_3 & = & 0 \end{array} \right\} \Rightarrow \begin{array}{rcl} \lambda_1 & = & -3c \\ \lambda_2 & = & -2c \\ \lambda_3 & = & c \end{array}, \quad c \in \mathbb{R}$$

So for $c = -1$, we have $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = -1$,

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Example

Let P be the vector space of polynomials. Show that the polynomials

$$\underline{x}_1 = 1 - t, \quad \underline{x}_2 = t^3 - 1, \quad \underline{x}_3 = t - t^3.$$

are linearly dependent vectors of P .

Solution

We can observe that

$$\underline{x}_1 + \underline{x}_2 + \underline{x}_3 = 1 - t + t^3 - 1 + t - t^3 = 0.$$

So

$$\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}, \quad \lambda_1 = \lambda_2 = \lambda_3 = 1 \neq 0$$

such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3 = \underline{0}.$$

Therefore these vectors are linearly dependent vectors of the space P .

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Example

Let $C[-\pi, \pi]$ be the vector space of the continuous functions which are defined on the interval $[a, b]$. Show that the functions

$$1, \sin x^2, \cos x^2$$

are linearly dependent vectors of the space $C[-\pi, \pi]$.

Solution

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that

$$\lambda_1 \cdot 1 + \lambda_2 \cdot \sin^2 x + \lambda_3 \cdot \cos^2 x = 0.$$

If we choose $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$, then there exists a non-zero solution of the above equation. Therefore these functions are linearly dependent vectors of the space $C[-\pi, \pi]$. ■

Example

Let $C^2[0,\pi/2]$ be the vector space of the twice differentiable functions which are defined on the interval $[a,b]$. Show that the functions

$$1, \sin x, \cos x$$

are linearly independent vectors of the space $C^2[0,\pi/2]$.

Solution

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that $\lambda_1 \cdot 1 + \lambda_2 \cdot \sin x + \lambda_3 \cdot \cos x = 0$.

Then, if we differentiate twice the above relation, we obtain

$$\begin{aligned}\lambda_2 \cdot \cos x - \lambda_3 \cdot \sin x &= 0 \quad \text{and} \\ -\lambda_2 \cdot \sin x - \lambda_3 \cdot \cos x &= 0\end{aligned}$$

From the last two equations we get the system

$$\left. \begin{aligned}\lambda_2 \cdot \cos x - \lambda_3 \cdot \sin x &= 0 \\ -\lambda_2 \cdot \sin x - \lambda_3 \cdot \cos x &= 0\end{aligned} \right\} \Rightarrow \left. \begin{aligned}\lambda_2 \cdot \cos x \cdot \sin x - \lambda_3 \cdot \sin^2 x &= 0 \\ -\lambda_2 \cdot \sin x \cdot \cos x - \lambda_3 \cdot \cos^2 x &= 0\end{aligned} \right\} \Rightarrow -\lambda_3(\sin^2 x + \cos^2 x) = 0 \Rightarrow \lambda_3 = 0$$

Thus $\lambda_3 = 0$ and so $\lambda_2 = 0$ and $\lambda_1 = 0$.

Therefore these functions are linearly independent vectors of the space $C^2[0,\pi/2]$.

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Example

Let V be the vector subspace $\mathbb{R}^3$ which consists of the solutions of the equation

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

where $a_1, a_2, a_3 \neq 0$. Find the dimension of the subspace V .

Solution

Since $a_3 \neq 0$, $x_3 = -\frac{a_1}{a_3}x_1 - \frac{a_2}{a_3}x_2$. Thus

$$\begin{aligned} V &= \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid a_1x_1 + a_2x_2 + a_3x_3 = 0 \right\} = \left\{ \left(x_1, x_2, -\frac{a_1}{a_3}x_1 - \frac{a_2}{a_3}x_2 \right) \mid x_1, x_2 \in \mathbb{R} \right\} \\ &= \left\{ \left(x_1, 0, -\frac{a_1}{a_3}x_1 \right) + \left(0, x_2, -\frac{a_2}{a_3}x_2 \right) \mid x_1, x_2 \in \mathbb{R} \right\} = \left\{ x_1 \left(1, 0, -\frac{a_1}{a_3} \right) + x_2 \left(0, 1, -\frac{a_2}{a_3} \right) \mid x_1, x_2 \in \mathbb{R} \right\} \end{aligned}$$

The vectors $\left(1, 0, -\frac{a_1}{a_3} \right)$ and $\left(0, 1, -\frac{a_2}{a_3} \right)$ are linearly independent and span V .

Hence these vectors are a base of V and so $\dim V = 2$.

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Example

Let V be a vector subspace $\mathbb{R}^4$ which is spanned by the vectors

$$\underline{x}_1 = (2, 1, 3, 1), \quad \underline{x}_2 = (1, 2, 0, 1), \quad \underline{x}_3 = (-1, 1, -3, 0),$$

of $\mathbb{R}^4$. Find the dimension of the subspace V .

Solution

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \lambda_3 \underline{x}_3 = \underline{0} \Rightarrow \lambda_1(2, 1, 3, 1) + \lambda_2(1, 2, 0, 1) + \lambda_3(-1, 1, -3, 0) = \underline{0} \Rightarrow$$

$$\left. \begin{array}{rcl} 2\lambda_1 + \lambda_2 - \lambda_3 & = & 0 \\ \lambda_1 + 2\lambda_2 + \lambda_3 & = & 0 \\ 3\lambda_1 - 3\lambda_3 & = & 0 \\ \lambda_1 + \lambda_2 & = & 0 \end{array} \right\} \Rightarrow \left. \begin{array}{rcl} 3\lambda_1 + 3\lambda_2 & = & 0 \\ \lambda_1 & = & \lambda_3 \\ \lambda_1 + \lambda_2 & = & 0 \end{array} \right\} \Rightarrow \begin{array}{rcl} \lambda_1 & = & c \\ \lambda_2 & = & -c, \quad c \in \mathbb{R} \\ \lambda_3 & = & c \end{array}$$

Hence these vectors are linearly dependent.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that

$$\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 = \underline{0} \Rightarrow \lambda_1 (2, 1, 3, 1) + \lambda_2 (1, 2, 0, 1) = \underline{0} \Rightarrow$$

$$\left. \begin{array}{rcl} 2\lambda_1 + \lambda_2 & = & 0 \\ \lambda_1 + 2\lambda_2 & = & 0 \\ 3\lambda_1 & = & 0 \\ \lambda_1 + \lambda_2 & = & 0 \end{array} \right\} \Rightarrow \begin{array}{rcl} \lambda_1 & = & 0 \\ \lambda_2 & = & 0 \end{array}$$

Hence these vectors are linearly independent and moreover span V . Therefore $\{\underline{x}_1, \underline{x}_2\}$ is a base of V and so $\dim V = 2$. ■