

LECTURE 10:

**GRAM-SCHIMDT
ORTHOGONALIZATION PROCESS**

Proposition

Suppose that the vector $\underline{b}$ can be written as a linear combination of the non-zero vectors

$\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$.

Then it holds that

$$\underline{b} = \sum_{k=1}^n \frac{\langle \underline{b}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \underline{a}_k.$$

Theorem (Gram-Schmidt orthogonalization process)

Let V be an inner product space and $\{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ be a set of n linearly independent vectors of V .

Then we find an orthogonal basis $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ that spans the same subspace W which is spanned by the set of vectors $\{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$.

Proof

In order to construct a new (orthogonal) basis $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$, we apply the Gram-Schmidt orthogonalization process.

At first, we choose $\underline{a}_1 = \underline{b}_1$. Assume that $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_m$ (where $1 \leq m < n$) are orthogonal and they are a basis for the space W , which is spanned by the vectors $\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m$. Then we can find the next vector $\underline{a}_{m+1}$ that is orthogonal to these vectors, as follows:

Let

$$\underline{a}_{m+1} = \underline{b}_{m+1} - \sum_{k=1}^m \frac{\langle \underline{b}_{m+1}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \underline{a}_k$$

Then $\underline{a}_{m+1} \neq \underline{0}$, because otherwise (i.e. $\underline{a}_{m+1} = \underline{0}$) the $\underline{b}_{m+1}$ will be a linear combination of $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_m$, which is a contradiction, since $\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m$ are linearly independent vectors.

So, for $j \in \{1, 2, \dots, m\}$, we obtain that

$$\begin{aligned}
 \langle \underline{a}_{m+1}, \underline{a}_j \rangle &= \langle \underline{b}_{m+1} - \sum_{k=1}^n \frac{\langle \underline{b}_{m+1}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \underline{a}_k, \underline{a}_j \rangle \\
 &= \langle \underline{b}_{m+1}, \underline{a}_j \rangle - \sum_{k=1}^n \frac{\langle \underline{b}_{m+1}, \underline{a}_k \rangle}{\|\underline{a}_k\|^2} \langle \underline{a}_k, \underline{a}_j \rangle \\
 &= \langle \underline{b}_{m+1}, \underline{a}_j \rangle - \langle \frac{\langle \underline{b}_{m+1}, \underline{a}_j \rangle}{\|\underline{a}_j\|^2} \underline{a}_j, \underline{a}_j \rangle \\
 &= \langle \underline{b}_{m+1}, \underline{a}_j \rangle - \langle \underline{b}_{m+1}, \underline{a}_j \rangle = 0
 \end{aligned}$$

Therefore $\underline{a}_{m+1} \perp \underline{a}_j$, for every $j \in \{1, 2, \dots, n\}$.

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Lecture 10:

Gram-Schmidt orthogonalization process

Example

Consider the linearly independent set of vectors $\{\underline{b}_1, \underline{b}_2\} = \{(3,0), (2,-2)\}$ of $\mathbb{R}^2$, which spans a vector subspace W . We choose $\underline{a}_1 = \underline{b}_1 = (3,0)$. Then

$$\begin{aligned}\underline{a}_2 &= \underline{b}_2 - \frac{\langle \underline{b}_2, \underline{a}_1 \rangle}{\|\underline{a}_1\|^2} \underline{a}_1 = (2,-2) - \frac{\langle (2,-2), (3,0) \rangle}{\|(3,0)\|^2} (3,0) \\ &= (2,-2) - \frac{6}{9} (3,0) = (2,-2) - (2,0) = (0,-2).\end{aligned}$$

The new (orthogonal) base of W will be

$$\{\underline{a}_1, \underline{a}_2\} = \{(3,0), (0,-2)\}.$$

Furthermore

$$\underline{a}_1' = \frac{\underline{a}_1}{\|\underline{a}_1\|} = \frac{(3,0)}{3} = (1,0) \quad \text{and} \quad \underline{a}_2' = \frac{\underline{a}_2}{\|\underline{a}_2\|} = \frac{(0,-2)}{2} = (0,-1).$$

Therefore the orthonormal base of W will be $\{\underline{a}_1', \underline{a}_2'\} = \{(1,0), (0,-1)\}$

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Example

Consider the linearly independent set of vectors $\{\underline{b}_1, \underline{b}_2, \underline{b}_3\} = \{(3, 0, 4), (-1, 0, 7), (2, 3, 11)\}$ of $\mathbb{R}^3$, which spans a vector subspace W .

We choose $\underline{a}_1 = \underline{b}_1 = (3, 0)$. Then

$$\begin{aligned}\underline{a}_2 &= \underline{b}_2 - \frac{\langle \underline{b}_2, \underline{a}_1 \rangle}{\|\underline{a}_1\|^2} \underline{a}_1 \\ &= (-1, 0, 7) - \frac{\langle (-1, 0, 7), (3, 0, 4) \rangle}{\|(3, 0, 4)\|^2} (3, 0, 4) \\ &= (-1, 0, 7) - \frac{25}{25} (3, 0, 4) = (-1, 0, 7) - (3, 0, 4) = (-4, 0, 3).\end{aligned}$$

Moreover

$$\begin{aligned}\underline{a}_3 &= \underline{b}_3 - \frac{\langle \underline{b}_3, \underline{a}_1 \rangle}{\|\underline{a}_1\|^2} \underline{a}_1 - \frac{\langle \underline{b}_3, \underline{a}_2 \rangle}{\|\underline{a}_2\|^2} \underline{a}_2 \\ &= (2, 3, 11) - \frac{\langle (2, 3, 11), (3, 0, 4) \rangle}{\|(3, 0, 4)\|^2} (3, 0, 4) - \frac{\langle (2, 3, 11), (-4, 0, 3) \rangle}{\|(-4, 0, 3)\|^2} (-4, 0, 3) \\ &= (2, 3, 11) - \frac{50}{25} (3, 0, 4) - \frac{25}{25} (-4, 0, 3) \\ &= (2, 3, 11) - (6, 0, 8) - (-4, 0, 3) \\ &= (0, 3, 0).\end{aligned}$$

The new (orthogonal) base of W is

$$\{\underline{a}_1, \underline{a}_2, \underline{a}_3\} = \{(3, 0, 4), (-4, 0, 3), (0, 3, 0)\}.$$

Furthermore

$$\underline{a}_1' = \frac{\underline{a}_1}{\|\underline{a}_1\|} = \frac{(3, 0, 4)}{5} = \left(\frac{3}{5}, 0, \frac{4}{5}\right),$$

$$\underline{a}_2' = \frac{\underline{a}_2}{\|\underline{a}_2\|} = \frac{(-4, 0, 3)}{5} = \left(-\frac{4}{5}, 0, \frac{3}{5}\right),$$

$$\underline{a}_3' = \frac{\underline{a}_3}{\|\underline{a}_3\|} = \frac{(0, 3, 0)}{3} = (0, 1, 0)$$

Therefore the orthonormal base of W is

$$\{\underline{a}_1', \underline{a}_2', \underline{a}_3'\} = \left\{ \left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0, 1, 0) \right\}.$$

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