

LECTURE 3:

DIAGONALIZATION - SPECIAL CASES

Let A be a 2×2 square matrix and λ_1, λ_2 its eigenvalues.

Then there exists an invertible matrix T such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}, \text{ if } \lambda_1 \neq \lambda_2 \in \mathbb{R},$$

that is if there exist two real eigenvalues with multiplicity 1.

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}, \text{ if } \lambda_1 = \lambda_2 \in \mathbb{R},$$

that is if there exist two real equal eigenvalues with multiplicity 2.

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \Rightarrow A = T \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} T^{-1}, \text{ if } \lambda_{1,2} = \alpha \pm \beta i \in \mathbb{C}, \quad \alpha, \beta \in \mathbb{R},$$

that is if there exist two conjugate complex eigenvalues.

We examine the following cases.

Case 1: Let A has two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$.

Then the matrix A will be written as follows

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

where $T = (\vec{x} \mid \vec{y})$ is a 2×2 square matrix and

$\vec{x}$ is an eigenvector which corresponds to the eigenvalue λ_1 and

$\vec{y}$ is an eigenvector which corresponds to the eigenvalue λ_2 .

Case 2: Let A has two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 = \lambda_2$.

Then matrix A will be written as previously

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} T^{-1}$$

if we can find two linearly independent eigenvectors which correspond to the eigenvalue $\lambda_1 = \lambda_2$

Otherwise, the matrix A can be written as follows

$$A = T \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$$

where $T = (\vec{x} \mid \vec{y})$ is a 2×2 square matrix and

$\vec{x}$ is an eigenvector which corresponds to the eigenvalue λ_1 and

$\vec{y}$ is an eigenvector which can be found by solving the system $[A - \lambda_1 I] \vec{y} = \vec{x}$.

Case 3: Let A has two eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\lambda_{1,2} = \alpha \pm \beta i$

Then matrix A will be written as follows

$$A = T \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} T^{-1}$$

where $T = (\vec{x} \mid \vec{y})$ is a 2×2 square matrix and $\vec{x}, \vec{y} \in \mathbb{R}^2$ are such that $\vec{z} = \vec{x} + \vec{y}i$ is an eigenvector which corresponds to the eigenvalue $\lambda_{1,2} = \alpha \pm \beta i$.

- Computation of A^n , $n \in \mathbb{N}$, when A in a 2×2 square matrix

Case 1: Let $A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}$.

Then $A^n = T \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}$, $n \in \mathbb{N}$.

Case 2: Let $A = T \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} T^{-1}$.

Then $A^n = T \begin{pmatrix} \lambda_1^n & n\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix} T^{-1}$, $n \in \mathbb{N}$.

Case 3: Let $A = T \begin{pmatrix} \alpha & \beta \\ -\beta & a \end{pmatrix} T^{-1}$, where $\alpha, \beta \in \mathbb{R}$ are such that $\lambda_{1,2} = \alpha \pm \beta i \in \mathbb{C}$

If ω is such that $\cos \omega = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$ and $\sin \omega = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$, then

$$\begin{aligned} \lambda_1 = \alpha + \beta i &= |\lambda_1| \left(\frac{\alpha}{|\lambda_1|} + \frac{\beta}{|\lambda_1|} i \right) = \sqrt{\alpha^2 + \beta^2} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} i \right) = \\ &= \sqrt{\alpha^2 + \beta^2} (\cos \omega + i \sin \omega) \end{aligned}$$

Then

$$B = \begin{bmatrix} \alpha & \beta \\ -\beta & a \end{bmatrix} = \sqrt{\alpha^2 + \beta^2} \begin{bmatrix} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} & \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \\ \frac{-\beta}{\sqrt{\alpha^2 + \beta^2}} & \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \end{bmatrix} = \sqrt{\alpha^2 + \beta^2} \begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix}$$

Hence

$$A^n = (\alpha^2 + \beta^2)^{n/2} T \begin{bmatrix} \cos(n\omega) & \sin(n\omega) \\ -\sin(n\omega) & \cos(n\omega) \end{bmatrix} T^{-1}.$$

Example

Consider the matrix

$$A = \begin{pmatrix} 8/3 & 1/3 \\ -4/3 & 4/3 \end{pmatrix}.$$

Compute the matrix A^n .

Solution:

From the characteristic equation

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 8/3 - \lambda & 1/3 \\ -4/3 & 4/3 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

we obtain the eigenvalue $\lambda_1 = 2$ with algebraic multiplicity $m_1 = 2$.

We will find the corresponding eigenvectors.

We consider the system $[A - 2I]\vec{x} = \vec{0}$, that is

$$[A - 2I]\vec{x} = \vec{0} \Rightarrow \begin{pmatrix} 2/3 & 1/3 \\ -4/3 & -2/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to $2x_1 + x_2 = 0$. We set $x_1 = c$, $c \in \mathbb{R}$.

Then

$$\vec{x} = (c \quad -2c)', \quad c \in \mathbb{R}.$$

We select $c = 1$, i.e.

$$\vec{x} = (1 \quad -2)$$

We consider the system $[A - 2I]\vec{x} = \vec{0}$, that is

$$[A - 2I]\vec{y} = \vec{x} \Rightarrow \begin{pmatrix} 2/3 & 1/3 \\ -4/3 & -2/3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

which leads to $\frac{2}{3}y_1 + \frac{1}{3}y_2 = 1 \Rightarrow 2y_1 + y_2 = 3$. We set $y_1 = c$, $c \in \mathbb{R}$.

Then

$$\vec{y} = (c \quad 3 - 2c)' \quad c \in \mathbb{R}.$$

We select $c = 0$, i.e.

$$\vec{y} = (0 \quad 3)$$

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_1 = 2$ are

$$\begin{aligned} \vec{x} &= (1 \quad -2) \\ \vec{y} &= (0 \quad 3) \end{aligned}$$

Hence

$$T = (\vec{x} \mid \vec{y}) = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$$

and so

$$T^{-1} = \frac{1}{3} \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/3 & 1/3 \end{pmatrix}.$$

Therefore

$$A^n = T \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix} T^{-1}, \quad n \in \mathbb{N}.$$

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Example

Consider the matrix

$$A = \begin{pmatrix} -2 & 1 \\ -7 & 3 \end{pmatrix}.$$

Compute the matrix A^n .

Solution:

From the characteristic equation

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2 - \lambda & 1 \\ -7 & 3 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \lambda + 1 = 0$$

we obtain the eigenvalues $\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$.

We will find the corresponding eigenvectors.

We consider the system

$$\left[A - \frac{1+i\sqrt{3}}{2} I \right] \vec{z} = \vec{0}, \Rightarrow \begin{pmatrix} -\frac{5+i\sqrt{3}}{2} & 1 \\ -7 & \frac{5-i\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to $-\frac{5+i\sqrt{3}}{2}z_1 + z_2 = 0$. We set $z_1 = c$, $c \in \mathbb{R}$.

Then

$$\vec{z} = \begin{pmatrix} c & \frac{5+i\sqrt{3}}{2}c \end{pmatrix}', \quad c \in \mathbb{R}.$$

We select $c = 1$, i.e.

$$\vec{z} = \begin{pmatrix} 1 \\ \frac{5+i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

Hence

$$T = \begin{pmatrix} 1 & 0 \\ 5/2 & \sqrt{3}/2 \end{pmatrix}$$

and so

$$T^{-1} = \frac{1}{\sqrt{3}/2} \begin{pmatrix} \sqrt{3}/2 & 0 \\ -5/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5/\sqrt{3} & 2/\sqrt{3} \end{pmatrix}.$$

We express the complex eigenvalue $\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$
to the trigonometric form $\lambda_{1,2} = \cos \omega \pm i \sin \omega$

$$\text{where } \cos \omega = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} = \frac{1/2}{\sqrt{(1/2)^2 + (\sqrt{3}/2)^2}} = \frac{1}{2} \Rightarrow \omega = \frac{\pi}{3}$$

$$\text{and } \sin \omega = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} = \frac{\sqrt{3}/2}{\sqrt{(1/2)^2 + (\sqrt{3}/2)^2}} = \frac{\sqrt{3}}{2} \Rightarrow \omega = \frac{\pi}{3}.$$

Therefore

$$A^n = (\alpha^2 + \beta^2)^{n/2} T \begin{bmatrix} \cos(n\omega) & \sin(n\omega) \\ -\sin(n\omega) & \cos(n\omega) \end{bmatrix} T^{-1} \Rightarrow A^n = T \begin{bmatrix} \cos \frac{n\pi}{3} & \sin \frac{n\pi}{3} \\ -\sin \frac{n\pi}{3} & \cos \frac{n\pi}{3} \end{bmatrix} T^{-1}, n \in \mathbb{N}.$$

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