

LECTURE 2: | MATRIX DIAGONALIZATION

Proposition

We consider the following $n \times n$ square matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

where a_{ij} , $1 \leq i \leq n$, $1 \leq j \leq n$ are real constants.

Assume that A has the discrete eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Then there exists an invertible matrix T such that

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \text{diag}(\lambda_1, \dots, \lambda_n).$$

The i -th column of the matrix T is an eigenvector which corresponds to the eigenvalue λ_i .

Application

Applying previous proposition we can compute the matrix A^m , $m \in \mathbb{N}$, where A is an $n \times n$ matrix which has discrete eigenvalues, Indeed, from the previous proposition we obtain that

$$B^m = (T^{-1}AT)^m = T^{-1}ATT^{-1}AT \dots T^{-1}AT = T^{-1}A^mT.$$

Therefore

$$A^m = TB^mT^{-1}.$$

But, since it is known that

$$B^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix}.$$

we get that

$$A^m = T \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^m \end{pmatrix} T^{-1}.$$

Example 2.1

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$

- a) Find the eigenvalues and the eigenvectors of the matrix A .
- b) Find a matrix T so that the matrix $T^{-1}AT$ be diagonal.
- c) Compute the matrix A^m , $m \in \mathbb{N}$.

Solution:

- a) We create the matrix

$$(A - \lambda I) = \begin{pmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{pmatrix}.$$

Then from the characteristic equation

$$|A - \lambda I| = (1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0$$

we obtain the eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$.

We will find now the eigenvectors which correspond to these eigenvalues.

Let $\lambda_1 = 2$. We consider the homogenous system $(A - 2I)\vec{x} = \vec{0}$, that is

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to $x_1 + x_2 = 0$. We set $x_1 = c$, $c \in \mathbb{R}$.

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_1 = 2$, are of the form

$$\vec{x} = (c \quad -c)'$$

We select the eigenvector

$$\vec{x} = (1 \quad -1)'$$

Let $\lambda_2 = 3$. We consider the homogenous system $(A - 3I)\vec{y} = \vec{0}$, that is

$$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which leads to $2y_1 + y_2 = 0$. We set $y_1 = c$, $c \in \mathbb{R}$.

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_2 = 3$, are of the form

$$\vec{y} = (c \quad -2c)'$$

We select the eigenvector

$$\vec{x} = (1 \quad -2)'$$

b) We define the matrix

$$T = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

We find its inverse matrix T^{-1} that is

$$T^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Hence

$$B = T^{-1}AT = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

c) We have

$$B^m = T^{-1}A^mT = \begin{pmatrix} 2^m & 0 \\ 0 & 3^m \end{pmatrix}$$

Therefore

$$A^m = T \begin{pmatrix} 2^m & 0 \\ 0 & 3^m \end{pmatrix} T^{-1}, \quad m \in \mathbb{N}.$$

Example 2.2

Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ -4 & -2 & -6 \\ 2 & 2 & 5 \end{pmatrix}.$$

- a) Find the eigenvalues and the eigenvectors of the matrix A .
- b) Find a matrix T so that the matrix $T^{-1}AT$ be diagonal.
- c) Compute the matrix A^m , $m \in \mathbb{N}$.

Solution:

- a) We create the matrix

$$(A - \lambda I) = \begin{pmatrix} 3 - \lambda & 1 & 2 \\ -4 & -2 - \lambda & -6 \\ 2 & 2 & 5 - \lambda \end{pmatrix}.$$

Then from the characteristic equation

$$\begin{aligned} |A - \lambda I| &= (3 - \lambda)((-2 - \lambda)(5 - \lambda) + 12) - \\ &\quad -(-20 + 4\lambda + 12) \\ &\quad + 2(-8 + 4 + 2\lambda) = \\ &= (3 - \lambda)(\lambda - 1)(\lambda - 2) = 0. \end{aligned}$$

we obtain the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

We will find now the eigenvectors which correspond to these eigenvalues.

Let $\lambda_1 = 1$. We consider the homogenous system $(A - I)\vec{x} = \vec{0}$, that is

$$\begin{pmatrix} 2 & 1 & 2 \\ -4 & -3 & -6 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We set $x_3 = c$, $c \in \mathbb{R}$. Then we get the system

$$\left. \begin{array}{l} 2x_1 + x_2 = -2c \\ x_1 + x_2 = -2c \end{array} \right\} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = -2c \end{array}$$

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_1 = 1$, are of the form

$$\vec{x} = c(0 \quad -2 \quad 1)'$$

where $c \in \mathbb{R}$. We select the eigenvector

$$\vec{x} = (0 \quad -2 \quad 1)'$$

Let $\lambda_2 = 2$. We consider the homogenous system $(A - 2I)\vec{y} = \vec{0}$, that is

$$\begin{pmatrix} 1 & 1 & 2 \\ -4 & -4 & -6 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We set $y_2 = c$, $c \in \mathbb{R}$. Then we get the system

$$\left. \begin{array}{l} y_1 + 2y_3 = -c \\ 2y_1 + 3y_3 = -2c \end{array} \right\} \Rightarrow \begin{array}{l} y_1 = -c \\ y_3 = 0 \end{array}$$

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_2 = 2$, are of the form

$$\vec{y} = c(-1 \ 1 \ 0)'.$$

where $c \in \mathbb{R}$. We select the eigenvector

$$\vec{y} = (-1 \ 1 \ 0)'.$$

Let $\lambda_3 = 3$. We consider the homogenous system $(A - 3I)\vec{x} = \vec{0}$, that is

$$\begin{pmatrix} 0 & 1 & 2 \\ -4 & -5 & -6 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We set $z_3 = c$, $c \in \mathbb{R}$. Then we get the system

$$\left. \begin{array}{l} z_2 = -2c \\ z_1 + z_2 = -c \end{array} \right\} \Rightarrow \begin{array}{l} z_1 = c \\ z_2 = -2c \end{array}$$

Therefore the eigenvectors which correspond to the eigenvalue $\lambda_3 = 3$, are of the form

$$\vec{z} = c(1 \quad -2 \quad 1)'.$$

where $c \in \mathbb{R}$. We select the eigenvector

$$\vec{z} = (1 \quad -2 \quad 1)'.$$

b) We define the matrix

$$T = \begin{pmatrix} 0 & -1 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}.$$

We find its inverse matrix T^{-1} , using row transformations to the following augmented matrix

$$(T \mid I) \sim \left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{H_{13}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array} \right) \xrightarrow{H_{21}(2)} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{array} \right) \xrightarrow{H_{32}(1)} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array} \right) \xrightarrow{H_{13}(-1)} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array} \right) \sim (I \mid T^{-1}).$$

Thus

$$T^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}..$$

Hence

$$B = T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

c) We have

$$B^m = T^{-1}A^mT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 3^m \end{pmatrix}$$

Therefore

$$A^m = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^m & 0 \\ 0 & 0 & 3^m \end{pmatrix} T^{-1}, \quad m \in \mathbb{N}.$$