

LECTURE 1:

EIGENVALUES AND EIGENVECTORS

We consider the following $n \times n$ square matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

where a_{ij} , $1 \leq i \leq n$, $1 \leq j \leq n$ are real constants. We call **characteristic equation** of the matrix A the equation

$$|A - \lambda I| = 0,$$

where I is the $n \times n$ identity matrix and $\lambda \in \mathbb{R}$. The roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the characteristic equation are called **characteristic values** or **eigenvalues** of the matrix A . Let λ_i be an eigenvalue of A . We consider the

$$(A - \lambda_i I)\vec{x} = \vec{0}$$

which has non-zero solutions since $|A - \lambda_i I| = 0$. Then any solution $\vec{x}$ of this homogenous system will be called **characteristic vector** or **eigenvector** which corresponds to the eigenvalue λ_i . Each eigenvector span a space which is called **eigenspace**.

Proposition

The matrices A και A^T have the same eigenvalues (A^T is the transpose matrix of A).

Proposition

If $\lambda_1, \lambda_2, \dots, \lambda_m$ are eigenvalues of the matrix A and k is a real constant, then the values $k\lambda_1, k\lambda_2, \dots, k\lambda_m$ will be eigenvalues of the matrix kA .

Definition:

A matrix A is called orthogonal if $AA' = I$.

Proposition

If λ is an eigenvalue of an orthogonal matrix A , then the value $1/\lambda$ will be eigenvalue of the matrix A .

Example 1.1

Consider the matrix

$$A = \begin{pmatrix} 6 & -3 \\ 3 & 12 \end{pmatrix}$$

Find:

the eigenvalues,

the eigenvectors and

the dimensions of the eigenspaces of A .

Solution: We consider the matrix

$$(A - \lambda I) = \begin{pmatrix} 6 - \lambda & -3 \\ 3 & 12 - \lambda \end{pmatrix}.$$

Then $|A - \lambda I| = (6 - \lambda)(12 - \lambda) + 9 = \lambda^2 - 18\lambda + 81 = (\lambda - 9)^2$.

From the characteristic equation $|A - \lambda I| = (\lambda - 9)^2 = 0$,

we obtain the eigenvalue $\lambda = 9$, which has multiplicity 2.

We will find now the eigenvector and the dimension of the eigenspace which correspond to this eigenvalue.

We consider the homogenous system $(A - 9I)\vec{x} = \vec{0}$ that is the system

$$\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus we get $x_1 + x_2 = 0$. We set $x_2 = c$, $c \in \mathbb{R}$. Therefore the eigenvector $\vec{x}$ which corresponds to the $\lambda = 9$ is of form $\vec{x} = (-c \ c)'$ where $c \in \mathbb{R}$.

b) Since $\vec{x} = c(-1 \ 1)'$ it is obvious that the dimension of the eigenspace which correspond to this eigenvalue is equal to 1.

■

Example 1.2

Consider the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. Find:

the eigenvalues,
the eigenvectors and
the dimensions of the eigenspaces of A .

Solution: We consider the matrix

$$(A - \lambda I) = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}.$$

Then $|A - \lambda I| = (2 - \lambda)((3 - \lambda)(2 - \lambda) - 2) - 2(2 - \lambda - 1) + (2 - 3 + \lambda) = -(\lambda - 1)^2(\lambda - 5)$.

From the characteristic equation $|A - \lambda I| = -(\lambda - 1)^2(\lambda - 5) = 0$ we obtain the eigenvalues

$\lambda_1 = 1$, which has multiplicity 2,

$\lambda_2 = 5$, which has multiplicity 1.

We will find now the eigenvectors and the dimensions of the eigenspaces which correspond to these eigenvalues.

For $\lambda_1 = 1$ we have the homogenous system $(A - 1I)\vec{x} = \vec{0}$ that is the system

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus we get $x_1 + 2x_2 + x_3 = 0$. We set $x_2 = c$, $x_3 = d$, $c, d \in \mathbb{R}$.

Hence $x_1 = -2c - d$, $x_2 = c$, $x_3 = d$, $c, d \in \mathbb{R}$.

Therefore the eigenvector $\vec{x}$ which corresponds to the $\lambda = 1$ is of form

$$\vec{x} = (-2c - d \quad c \quad d)'$$

that is

$$\vec{x} = c(-2 \quad 1 \quad 0)' + d(-1 \quad 0 \quad 1)'$$

where $c, d \in \mathbb{R}$.

Thus the corresponding eigenspace is

$$V_1 = \{c(-2 \quad 1 \quad 0)' + d(-1 \quad 0 \quad 1)': c, d \in \mathbb{R}\}.$$

Since the eigenspace V_1 is spanned by two linearly independent vectors, the dimension of the eigenspace is equal to 2,

Lecture 1:

Eigenvalues and eigenvectors

For $\lambda_2 = 5$ we have the homogenous system $(A - 5I)\vec{x} = \vec{0}$ that is the system

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Applying the Gauss-Jordan elimination method we have

$$(A - 5I | \vec{0}) = \left(\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array} \right) \xrightarrow{H_{12}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 2 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array} \right) \xrightarrow{H_{21}(3)} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \end{array} \right)$$

$$\xrightarrow{H_2(-1/4)} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{array} \right) \xrightarrow{H_{12}(2)} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & -4 & 0 \end{array} \right) \xrightarrow{H_{32}(-4)} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

we get the system
$$\begin{cases} y_1 - y_3 = 0 \\ y_2 - y_3 = 0 \end{cases}$$

We set $y_2 = c$, $c \in \mathbb{R}$. Hence $y_1 = c$, $y_3 = c$, $c \in \mathbb{R}$

Therefore the eigenvector $\vec{y}$ which corresponds to the $\lambda_2 = 5$ is of form

$$\vec{y} = (c \quad c \quad c)', \quad c \in \mathbb{R}$$

Thus the corresponding eigenspace is

$$V_2 = \{c(1 \quad 1 \quad 1)': c \in \mathbb{R}\}.$$

Since the eigenspace V_2 is spanned by one vector, the dimension of the eigenspace is equal to 1. ■